

Baryon distribution amplitude: Large- N_c factorization, spin-flavor symmetry and soft-pion theorem

P.V. Pobylitsa

*Institute for Theoretical Physics II, Ruhr University Bochum, D-44780 Bochum, Germany
Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg, 188300, Russia*

The $1/N_c$ expansion for the baryon distribution amplitude is constructed in terms of a specially designed generating functional. At large N_c this functional shows exponential behavior. The exponential factor is universal for all low-lying baryons and baryon-meson scattering states. Simple factorization properties are established for the preexponential term. This factorization agrees with the large- N_c contracted $SU(2N_f)$ spin-flavor symmetry. The consistency of the factorization with the soft-pion theorem for the baryon distribution amplitude is explicitly checked. A relation between the generating functionals for the distribution amplitudes of the nucleon and the Δ resonance is derived.

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I. INTRODUCTION

A. Baryon distribution amplitude in large- N_c QCD

Baryon distribution amplitude $\Psi^B(x_1, x_2, x_3)$ carries information on the 3-quark component of the baryon in the infinite momentum frame. The variables x_k have the meaning of the momentum fraction of the baryon carried by the k th quark. The baryon distribution amplitude plays an important role in the QCD analysis of hard exclusive phenomena [1, 2, 3].

Perturbative QCD provides information about the distribution amplitude only in the asymptotic regime of large scales μ . At finite μ , the problem of the calculation of the baryon distribution amplitude belongs to nonperturbative physics. The amount of methods which allow us to go beyond the bounds of perturbative QCD (without leaving the solid ground of QCD) is rather small. The limit of the large number of colors N_c and the $1/N_c$ expansion [4, 5, 6, 7, 8] belong to this type of method. The problem of the calculation of the baryon distribution amplitude in the $1/N_c$ expansion was raised in Refs. [9, 10].

In QCD with N_c quark colors, the minimal quark component of the baryon contains N_c quarks so that the baryon distribution amplitude $\Psi^B(x_1, x_2, \dots, x_{N_c})$ depends on N_c variables x_k . It hardly makes sense to speak about the large- N_c asymptotic behavior of a function depending on N_c variables. In order to construct the $1/N_c$ expansion for the baryon distribution amplitude, one has to introduce an object which contains the same information as the distribution amplitude but allows for a systematic $1/N_c$ expansion. In Ref. [11] it was suggested to use the generating functional $\Phi_B(g)$ depending on an arbitrary “source” $g(x)$. In an oversimplified form its definition is

$$\Phi_B(g) \sim \int \Psi^B(x_1, x_2, \dots, x_{N_c}) g(x_1) g(x_2) \dots g(x_{N_c}) dx_1 dx_2 \dots dx_{N_c} . \quad (1.1)$$

The precise definition of $\Phi_B(g)$ requires more care about N_c -dependent normalization factors, spin and flavor indices, etc. (see Sec. II).

As was shown in Ref. [11], the functional $\Phi_B(g)$ has an exponential behavior at large N_c :

$$\Phi_B(g) \stackrel{N_c \rightarrow \infty}{\sim} \exp[N_c W(g)] . \quad (1.2)$$

Functional $W(g)$ has several important properties [11]:

- 1) The same functional $W(g)$ describes all low-lying baryons as well as baryon-meson scattering states [universality of $W(g)$].
- 2) Functional $W(g)$ obeys a nonlinear evolution equation which can be solved analytically in the asymptotic regime.
- 3) The problem of the diagonalization of the anomalous dimensions of the leading-twist baryon operators can be formulated (and solved analytically) in terms of the nonlinear evolution equation for the functional $W(g)$.

The *exponential* behavior controlled by the functional $W(g)$ is a peculiar feature of the problem of the baryon distribution amplitude in large- N_c QCD. The traditional methods of the $1/N_c$ expansion were developed for quantities with a *power* large- N_c behavior. In Ref. [11] the analysis of the exponential large- N_c behavior was based on the nonlinear evolution equation for the functional $W(g)$.

Although the approach based on the nonlinear evolution equation has allowed one to establish rather interesting properties of the $1/N_c$ expansion, this method has certain limitations: The evolution equation has a perturbative origin. One can inject the nonperturbative physics into the evolution equation as an initial condition at some normalization point but one cannot extract the nonperturbative physics from the evolution equation.

In this work we use nonperturbative tools for the analysis of the large- N_c structure of the generating functional $\Phi_B(g)$. The standard large- N_c methods developed for quantities with the power large- N_c behavior include a very important ingredient, so-called contracted $SU(2N_f)$ spin-flavor symmetry [12, 13, 14, 15, 16, 17]. In this paper, we will study manifestations of the spin-flavor symmetry in the $1/N_c$ expansion of the functional $\Phi_B(g)$. Another nonperturbative approach to the analysis of the large- N_c limit of $\Phi_B(g)$ is based on the soft-pion theorem.

As was mentioned above, the exponential behavior (1.2) is controlled by the universal functional $W(g)$. However, we also have the preexponential functional $A_B(g)$:

$$\Phi_B(g) \sim A_B(g) \exp[N_c W(g)] . \quad (1.3)$$

The functional $A_B(g)$ depends on the type of the baryon B . The structure of this dependence is one of the subjects of this paper.

B. Methods and results

In Ref. [11] the functionals $W(g)$ and $A_B(g)$ were computed in the asymptotic limit of a large normalization scale μ . However, in the general case of a finite scale μ , the problem of the calculation of the functionals $W(g)$ and $A_B(g)$ belongs to the same class of difficulty as solving large- N_c QCD.

One can introduce the baryon distribution amplitude not only for stable baryons but also for baryon resonances and for baryon-meson scattering states (including one baryon and an arbitrary amount of mesons). In some sense the functional $A_B(g)$ can be considered as a sort of special large- N_c representation for the baryon and baryon-meson states $|B\rangle$.

The large- N_c classification of baryon-meson scattering states $|B\rangle$ is well known [18, 19, 20, 21, 22, 23, 24]. Historically this problem was first studied in the framework of the Skyrme model [25, 26, 27] but from the very beginning it was obvious that the case of large- N_c QCD is similar. Usually the large- N_c analysis is applied to the scattering states containing one baryon and one meson. However, this construction can be easily extended to the case of multi-meson scattering states. Up to subtleties related to zero modes, one can represent these scattering states as

$$|B\rangle = \prod_k a_{\beta_k}^+ |B_0\rangle , \quad (1.4)$$

where the operators $a_{\beta_k}^+$ create scattering states for single mesons and $|B_0\rangle$ is the pure baryon state.

In this paper, we show that the functional $A_B(g)$ corresponding to the state (1.4) factorizes into a product of the “baryon ground state” functional $A_0(g)$ and functionals $\Xi_{\beta_k}(g)$ corresponding to the elementary $a_{\beta_k}^+$ excitations. The oversimplified form of this factorization is

$$A_B(g) = A_0(g) \prod_k \Xi_{\beta_k}(g) . \quad (1.5)$$

However, the real structure is more complicated because of zero modes. The full description of $O(N_c^0)$ excitations requires the introduction of extra degrees of freedom in addition to the a_{β}^+ excitations. For example, in the case of QCD with a $SU(2)$ isospin symmetry, the additional degrees of freedom are represented by a rotator whose states can be described by wave functions $\psi(U)$ depending on the $SU(2)$ matrices U . As a result, the baryon and baryon-meson states can be represented at large N_c as linear combinations of the states

$$|B'\rangle = \psi(U) \prod_k a_{\beta_k}^+ |B_0\rangle . \quad (1.6)$$

The appearance of this U degree of freedom leads to a modification of the naive factorization (1.5) of the functional $A_B(g)$. In this paper, we show that the presence of the factor $\psi(U)$ in expression (1.6) for the state $|B'\rangle$ has a very simple manifestation in the corresponding functional $A_{B'}(g)$ describing the distribution amplitude of the state B' :

$$A_{B'}(g) = \psi[Q(g)] A_0(g) \prod_k \Xi_{\beta_k}(g) , \quad (1.7)$$

Here $Q(g)$ is some 2×2 matrix functional of g . This functional is “universal” in the sense that it does not depend on the state B' . The functional $Q(g)$ plays the same role for functionals $A_{B'}(g)$ in Eq. (1.7) as the matrix U for the states (1.6). However, there is one important difference. Although matrix U is unitary in Eq. (1.6), the matrix functional $Q(g)$ is not unitary but still belongs to the group $SL(2, C)$. Therefore $\psi[Q(g)]$ should be understood in terms of the analytical continuation of $\psi(U)$ from $SU(2)$ to $SL(2, C)$. This analytical continuation makes no problems, since in practical work the role of functions $\psi(U)$ is played by the $SU(2)$ Wigner D functions, whose complexification for the $SL(2, C)$ case is straightforward.

We stress that Eqs. (1.6) and (1.7) expressing the correspondence between the states $|B'\rangle$ and functionals $A_{B'}(g)$ are written in an oversimplified form. The precise form of these equations can be found in Sec. IV.

Now let us turn to the methods which allow us to derive the factorization property (1.7). As is well known, large- N_c QCD is not solved. The “derivations” of most of the results in large- N_c QCD are based on the extrapolation of the perturbative large- N_c counting to the nonperturbative region, on various models (naive quark model, Skyrme model, chiral quark soliton model, etc.) and on checks of the consistency with current algebra, soft-pion theorems, etc.

Our argumentation in favor of the factorization (1.7) also cannot be considered as an impeccable final proof. Nevertheless we can present a rather wide set of arguments. Some of them have been already suggested in Ref. [11]:

- Factorization (1.7) is consistent with the evolution equation for the functional $A_B(g)$.
- This factorization was observed in the asymptotic regime of large normalization scales μ , where the functional $A_B(g)$ can be computed analytically.
- The factorization of the type (1.7) can be seen in the naive quark model.

In Ref. [28] factorization (1.7) was studied in large- N_c models with the following results:

- The factorization was checked using the reduction of the large- N_c problem to the analysis of effective classical dynamical systems [29, 30].
- The connection was established between this factorization and the traditional methods of the $1/N_c$ expansion based on the Hartree and RPA (random phase approximation) equations [31, 32].

In this paper, we describe a new group of arguments:

- Factorization (1.7) is consistent with the large- N_c contracted $SU(2N_F)$ spin-flavor symmetry.
- This factorization is consistent with the soft-pion theorem for the distribution amplitude of pion-baryon scattering states [33].

As is well known, in QCD with the $SU(N_f)$ flavor symmetry, the large- N_c limit leads to the additional so-called contracted $SU(2N_f)$ symmetry [12, 13, 14, 15, 16, 17]. The usual rotational group $SO(3) \sim SU(2)$ and the flavor group $SU(N_f)$ are described by the standard Lie algebra

$$[J_k, J_l] = i\varepsilon_{klm}J_m, \quad (1.8)$$

$$[T_a, T_b] = if_{abc}T_c, \quad (1.9)$$

$$[J_k, T_a] = 0 \quad (1.10)$$

with the angular momentum generators J_k and the flavor generators T_a . As usual, summation over repeated indices is implied.

In the large- N_c limit, new symmetry generators X_{ia} appear with the commutation relations

$$[T_a, X_{jb}] = if_{abc}X_{jc}, \quad (1.11)$$

$$[J_k, X_{lb}] = i\varepsilon_{klm}X_{mb}, \quad (1.12)$$

$$[X_{ia}, X_{jb}] = 0. \quad (1.13)$$

The Lie algebra (1.8)–(1.13) corresponds to the contracted $SU(2N_f)$ group. The appearance of this additional symmetry at large N_c is based on general arguments and our large- N_c construction for the generating functional $\Phi_B(g)$ must be compatible with this symmetry. In this paper, we show how the contracted $SU(2N_f)$ spin-flavor group is realized in the space of functionals $A_B(g)$.

Another consistency check studied in this paper is based on the soft-pion theorem. Let us consider the state $|B\pi\rangle$ containing baryon B and pion π . If pion π is soft (in the rest frame of B), then the soft-pion theorem allows us to reduce the matrix elements of the state $|B\pi\rangle$ to the matrix elements of the state $|B\rangle$ without the soft pion. In

particular, the generating functional $\Phi_{B\pi}(g)$ describing the distribution amplitude of the state $|B\pi\rangle$ can be expressed via the generating functional $\Phi_B(g)$ of the pure baryon state $|B\rangle$. Therefore the soft-pion theorem imposes certain constraints on functionals $A_B(g)$ and $A_{B\pi}(g)$. In this paper, we show that these constraints are compatible with the large- N_c factorization properties of the functionals $A_B(g)$.

The structure of the paper is as follows. In Sec. II we give a precise definition of the functionals $\Phi_B(g)$, $A_B(g)$ and briefly describe the main results of Ref. [11], including the universality of $W(g)$ and the evolution equations for functionals $W(g)$ and $A_B(g)$. Section III contains the description of the standard picture of the large- N_c baryons and baryon-meson scattering states. The results considered in Sec. III are well known. We concentrate in Sec. III only on those facts which are needed for our analysis of the functional $A_B(g)$, including the large- N_c spin-flavor symmetry [12, 13, 14, 15, 16, 17] and the general properties of the baryon-meson scattering at large N_c [18, 19, 20, 21, 22, 23, 24]. In Sec. IV we turn to the factorization properties of the functional $A_B(g)$ and show that these properties are completely consistent with the large- N_c spin-flavor symmetry. As a sort of “practical application” of the general factorization equations, we derive an expression for the functional $A^{(\Delta)}(g)$ corresponding to the Δ resonance via the nucleon functional $A^{(N)}(g)$. In Sec. V we obtain the large- N_c generalization of the soft-pion theorem for the distribution amplitude of a near threshold pion-nucleon state [33] and show that the factorized structure of the functional $A_B(g)$ is also consistent with this soft-pion theorem.

II. BARYON DISTRIBUTION AMPLITUDE AT LARGE N_c

A. Definition

The baryon distribution amplitude Ψ^B is defined as a transition matrix element of a product of N_c light-cone quark fields $\chi_{f_k s_k c_k}(x_k)$ taken between the vacuum and the baryon B with momentum P :

$$\begin{aligned} \Psi_{(f_1 s_1)(f_2 s_2)\dots(f_{N_c} s_{N_c})}^B(x_1, x_2, \dots, x_{N_c}) \\ = \frac{1}{N_c!} \varepsilon_{c_1 c_2 \dots c_{N_c}} \langle 0 | \chi_{c_1 f_1 s_1}(x_1) \chi_{c_2 f_2 s_2}(x_2) \dots \chi_{c_{N_c} f_{N_c} s_{N_c}}(x_{N_c}) | B(P) \rangle. \end{aligned} \quad (2.1)$$

The indices f_k, s_k, c_k stand for the quark flavor, spin and color, respectively:

$$f_k = 1, 2, \dots, N_f, \quad (2.2)$$

$$s_k = \pm \frac{1}{2}, \quad (2.3)$$

$$c = 1, 2, \dots, N_c. \quad (2.4)$$

The tensor $\varepsilon_{c_1 c_2 \dots c_{N_c}}$ is used for the antisymmetrization in color. The light-cone quark operators $\chi_{f s c}(x)$ depending on the longitudinal quark momentum fraction x can be expressed in terms of the usual Dirac quark field q as follows

$$\chi_{c f s}(x) = (n P)^{1/2} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \bar{u}_{-s} q_{c f}(\lambda n) \exp[i\lambda x(n P)]. \quad (2.5)$$

Here n is an auxiliary light-cone vector. The Dirac spinors u_s associated with n are normalized by the condition

$$u_s \otimes \bar{u}_s = \frac{1}{2} (n \cdot \gamma) (1 - 2s\gamma_5) \quad \left(s = \pm \frac{1}{2} \right). \quad (2.6)$$

Strictly speaking, expression (2.1) for the baryon distribution amplitude is valid only for the light-cone gauge

$$(n \cdot A) = 0. \quad (2.7)$$

In other gauges, Eq. (2.1) must be modified by an insertion of Wilson lines.

Note that the distribution amplitude defined by Eq. (2.1) contains the momentum conserving delta function:

$$\Psi_{(f_1 s_1)(f_2 s_2)\dots(f_{N_c} s_{N_c})}^B(x_1, x_2, \dots, x_{N_c}) \sim \delta(x_1 + x_2 + \dots + x_{N_c} - 1). \quad (2.8)$$

B. Large- N_c limit

Although the definition (2.1) of the baryon distribution amplitude Ψ^B uses the auxiliary vector n [entering via the operators $\chi_{cfs}(x)$], the resulting function Ψ^B is independent of n . Usually one normalizes n by the condition $(nP) = 1$. However, at large N_c this normalization is not convenient. Indeed, at $N_c \rightarrow \infty$ we have $P = O(N_c)$ because of the $O(N_c)$ growth of the baryon mass. If we keep n fixed at large N_c , then (nP) will grow as $O(N_c)$:

$$n = O(N_c^0), \quad (nP) = O(N_c). \quad (2.9)$$

At large N_c because of the momentum conserving delta function (2.8) the distribution amplitude is concentrated at

$$x_k \sim 1/N_c. \quad (2.10)$$

Therefore, it is useful to introduce the new variables

$$y_k = N_c x_k \quad (2.11)$$

behaving as $O(N_c^0)$. Let us define the function

$$\Psi'^B_{(f_1 s_1)(f_2 s_2) \dots (f_{N_c} s_{N_c})}(y_1, y_2, \dots, y_{N_c}) = N_c^{-N_c/2} \Psi^B_{(f_1 s_1)(f_2 s_2) \dots (f_{N_c} s_{N_c})} \left(\frac{y_1}{N_c}, \frac{y_2}{N_c}, \dots, \frac{y_{N_c}}{N_c} \right) \quad (2.12)$$

which is more convenient for the transition to the large- N_c limit. The factor $N_c^{-N_c/2}$ is inserted here in order to compensate the N_c growth of the contribution of the kinematical factor $(nP)^{N_c/2}$ coming from the product of N_c fields χ (2.5).

Now we define

$$\begin{aligned} \Phi_B(g) &= \sum_{f_k s_k} \int_0^\infty dy_1 \int_0^\infty dy_2 \dots \int_0^\infty dy_{N_c} g_{f_1 s_1}(y_1) g_{f_2 s_2}(y_2) \dots g_{f_{N_c} s_{N_c}}(y_{N_c}) \\ &\times \Psi'^B_{(f_1 s_1)(f_2 s_2) \dots (f_{N_c} s_{N_c})}(y_1, y_2, \dots, y_{N_c}). \end{aligned} \quad (2.13)$$

The distribution amplitude can be expressed via the functional $\Phi_B(g)$:

$$\Psi^B_{(f_1 s_1)(f_2 s_2) \dots (f_{N_c} s_{N_c})} \left(\frac{y_1}{N_c}, \frac{y_2}{N_c}, \dots, \frac{y_{N_c}}{N_c} \right) = \frac{N_c^{N_c/2}}{N_c!} \left(\prod_{k=1}^{N_c} \frac{\delta}{\delta g_{f_k s_k}(y_k)} \right) \Phi_B(g). \quad (2.14)$$

Therefore, at finite N_c the functional $\Phi_B(g)$ and the distribution amplitude Ψ^B contain the same information.

As shown in Ref. [11], the large- N_c asymptotic behavior of $\Phi_B(g)$ is described by the representation

$$\Phi_B(g) = N_c^{\nu_B} A_B(g) \exp [N_c W(g)] [1 + O(N_c^{-1})]. \quad (2.15)$$

Here the naive expression (1.3) is modified by the factor $N_c^{\nu_B}$. The properties of this factor are studied in Sec. IV G.

C. Universality of $W(g)$ and factorization of $A_B(g)$

The functional $W(g)$ appearing in the large- N_c decomposition (2.15) is universal for all low-lying baryons [including resonances with $O(N_c^{-1})$ and $O(N_c^0)$ excitation energies]. The same functional $W(g)$ describes the exponential large- N_c behavior of the distribution amplitudes of baryon-meson scattering states containing one baryon and an arbitrary $O(N_c^0)$ amount of mesons. Moreover, this functional $W(g)$ also describes the large- N_c behavior of higher-twist distribution amplitudes with a finite $O(N_c^0)$ amount of extra insertions of color-singlet light-cone quark or gluon operators in addition to the product of N_c quark fields on the RHS of Eq. (2.1).

The universality of $W(g)$ was discussed in detail in Ref. [11]. In this paper, we concentrate on the properties of the preexponential functional $A_B(g)$ appearing in the large- N_c representation (2.15). In contrast to $W(g)$, the functional $A_B(g)$ depends on the baryon (or baryon-meson) state B . Perturbative QCD can provide us information on $A_B(g)$ only in the asymptotic limit of the high normalization scale $\mu \rightarrow \infty$. At finite μ , the functional $A_B(g)$ is determined by the nonperturbative dynamics of QCD and we cannot compute it from the first principles. However, the large- N_c limit allows us to establish certain important properties of $A_B(g)$. It is well known that the baryon resonances

with excitation energies $O(N_c^0)$ and the baryon-meson scattering states can be described in terms of “elementary excitations” [18, 19, 20, 21, 22, 23]. The complete description of the spectrum of the low-lying baryon and baryon-meson states also includes “zero modes” responsible for the “fine” $O(N_c^{-1})$ structure of the spectrum. In this paper, we show that the functional $A_B(g)$ corresponding to the baryon (baryon-meson) state B factorizes into a product of functionals associated with “elementary excitations” making the state B . The precise form of this factorization depends on the number of quark flavors N_f . In fact, because of the presence of zero modes, the functional $A_B(g)$ is a polynomial of elementary functionals rather than the simple product (1.7).

D. Evolution equation

The evolution equations for functionals $W(g)$ and $A_B(g)$ were derived and studied in detail in Ref. [11]. In this section we present a brief derivation of these equations which are needed for the analysis of the preexponential factors $A_B(g)$.

The dependence of the baryon distribution amplitude $\Psi^{B,\mu}(x_1 \dots x_{N_c})$ on the normalization point μ is described by the well known evolution equation [1]

$$\mu \frac{\partial}{\partial \mu} \Psi^{B,\mu}(x_1, \dots, x_{N_c}) = -\frac{N_c + 1}{2N_c} \frac{\alpha_s(\mu)}{\pi} \sum_{1 \leq i < j \leq N_c} \bar{K}_{ij} \Psi^{B,\mu}(x_1, \dots, x_{N_c}). \quad (2.16)$$

The evolution kernel has the simple form of a sum of “pair interactions” \bar{K}_{ij} between the i th and j th quark. In this paper, we use notation \bar{K} for the two-particle evolution kernel (instead of K as it was in Ref. [11]) in order to avoid confusion with the eigenvalues of the operator \mathcal{K} , which plays an important role in the spin-flavor symmetry classification of large- N_c baryons (Sec. III E).

This evolution equation can be rewritten in terms of the generating functional $\Phi_B^\mu(g)$ (2.13)

$$\mu \frac{\partial}{\partial \mu} \Phi_B^\mu(g) = -\frac{N_c + 1}{2N_c} \frac{\alpha_s(\mu)}{2\pi} \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\left(\frac{\delta}{\delta g} \otimes \frac{\delta}{\delta g} \right) \Phi_B^\mu(g) \right] \right\}. \quad (2.17)$$

Here we use the brief notation

$$\begin{aligned} & (g \otimes g) \cdot \bar{K} \cdot \left[\left(\frac{\delta}{\delta g} \otimes \frac{\delta}{\delta g} \right) \Phi_B^\mu(g) \right] \\ &= \int_0^\infty dy_1 \int_0^\infty dy_2 \sum_{f_1 f_2 s_1 s_2} g_{f_1 s_1}(y_1) g_{f_2 s_2}(y_2) \left[\bar{K} \frac{\delta}{\delta g_{f_1 s_1}(y_1)} \frac{\delta}{\delta g_{f_2 s_2}(y_2)} \Phi_B^\mu(g) \right]. \end{aligned} \quad (2.18)$$

Inserting the large- N_c decomposition (2.15) into Eq. (2.17), we obtain a nonlinear evolution equation for the functional $W_\mu(g)$

$$\mu \frac{\partial}{\partial \mu} W_\mu(g) = -a(\mu) \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\frac{\delta W_\mu(g)}{\delta g} \otimes \frac{\delta W_\mu(g)}{\delta g} \right] \right\}, \quad (2.19)$$

where

$$a(\mu) = \lim_{N_c \rightarrow \infty} \frac{\alpha_s(\mu) N_c}{4\pi}. \quad (2.20)$$

It is convenient to introduce the new variable t such that

$$dt = 2a(\mu) \frac{d\mu}{\mu}. \quad (2.21)$$

Then

$$\frac{\partial}{\partial t} W(g, t) = -\frac{1}{2} \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta W(g, t)}{\delta g} \right] \right\}. \quad (2.22)$$

In the next order of the $1/N_c$ expansion, we obtain the evolution equation for $A_B(g, t)$

$$\begin{aligned} \frac{\partial}{\partial t} \ln A_B(g, t) = & - \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta \ln A_B(g, t)}{\delta g} \right] \right\} \\ & + b(t) \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta W(g, t)}{\delta g} \right] \right\} \\ & - \frac{1}{2} \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\left(\frac{\delta}{\delta g} \otimes \frac{\delta}{\delta g} \right) W(g, t) \right] \right\}, \end{aligned} \quad (2.23)$$

where

$$b(t) = \frac{1}{2} \lim_{N_c \rightarrow \infty} \left\{ N_c \left[1 - \frac{N_c + 1}{4\pi} \frac{\alpha_s(\mu)}{a(\mu)} \right] \right\}. \quad (2.24)$$

III. BARYON-MESON SCATTERING STATES

In this section we give a brief description of well known properties of baryons and baryon-meson scattering states in large- N_c QCD. Later, these properties will be used in the analysis of the factorization of functionals $A_B(g)$.

A. $O(N_c^{-1})$ excited baryons

According to the standard picture of baryons in large- N_c QCD with $N_f = 2$ flavors, the lowest baryons have equal spin J and isospin T [26, 34]:

$$T = J = \begin{cases} \frac{1}{2}, \frac{3}{2}, \dots & \text{for odd } N_c \\ 0, 1, 2, \dots & \text{for even } N_c \end{cases}. \quad (3.1)$$

The masses of these baryons have a $1/N_c$ suppressed splitting:

$$M_{T=J} = d_1 N_c + d_0 + N_c^{-1} \left[d_{-1}^{(1)} + d_{-1}^{(2)} J(J+1) \right] + O(N_c^{-2}). \quad (3.2)$$

These baryons belong to the same representation of the spin-flavor symmetry group [12, 13, 14, 15, 16] which becomes asymptotically exact at large N_c . As a consequence, the functional $W(g)$ defined by Eq. (2.15) is the same for all low-lying baryons (3.1). The dependence on the type of baryons B appears only in the preexponential factor $A_B(g)$.

One should keep in mind that the excited baryons can be unstable with respect to decays into lower baryons and mesons. The precise spectrum of stable excited baryons is determined by the mass thresholds which depend on the quark masses. In the case of the $O(N_c^{-1})$ excited baryon resonances, their width has an additional $1/N_c$ suppression compared to $O(N_c^{-1})$ mass splittings. Therefore in the first orders of the $1/N_c$ expansion we can ignore the instability of these particles. In particular, in the leading order one can treat $O(N_c^{-1})$ excited baryon resonances as asymptotic states of the S matrix of the baryon-meson scattering.

B. $O(N_c^0)$ excited baryons and baryon-meson scattering states

Now we turn to the baryons with the $O(N_c^0)$ excitation energy. In this case the excitation energy is rather large and the problem of the instability of baryon resonances becomes more serious. Therefore the theory must be constructed in terms of baryon-meson scattering states. A general $O(N_c^0)$ excitation corresponds to a scattering state containing one baryon and an arbitrary (but finite compared to large N_c) amount of mesons. In principle, the general picture of the large- N_c limit does not exclude the existence of stable $O(N_c^0)$ excited baryons, although the real world seems to ignore this possibility.

Anyway, both baryon-meson scattering states [with the $O(N_c^0)$ meson energy in the c.m. frame] and “possible” stable $O(N_c^0)$ excited baryons can be described by a common large- N_c formalism [18, 19, 20, 21, 22, 23, 35]. The energy of these states is given by the formula

$$E = d_1 N_c + \left(d_0 + \sum_{\beta} \Omega_{\beta} n_{\beta} \right) + O(N_c^{-1}), \quad (3.3)$$

where $n_\beta = 0, 1, 2, \dots$ are integer numbers, and the Ω_β are N_c -independent coefficients.

In the case of stable $O(N_c^0)$ excited baryons, the energy E in Eq. (3.3) would correspond to the mass of the excited baryon. Eq. (3.3) represents the excitation corrections to the mass as a discrete set of “elementary” excitation energies Ω_β .

In the case of baryon-meson scattering states containing several mesons, the energies Ω_β should be interpreted as energies of single mesons. Obviously these single-meson energies Ω_β have a continuous spectrum. For the scattering states with the continuous spectrum of Ω_β , only the values $n_\beta = 0, 1$ in Eq. (3.3) have physical sense.

In both cases (stable excited baryons and baryon-meson scattering states), we can interpret the states (3.3) as a result of the action of creation operators a_β^+ on the state $|B_0\rangle$ corresponding to the lowest baryon

$$\prod_{\beta} \left(a_{\beta}^{+}\right)^{n_{\beta}} |B_0\rangle. \quad (3.4)$$

However, the full picture of the $O(N_c^0)$ excitations is more complicated. Quantum numbers n_β appearing in Eq. (3.3) do not fix the excited state completely. The complete description of these states must contain extra degrees of freedom in addition to the creation operators a_β^+ and additional quantum numbers associated with the new degrees of freedom. The dependence of the energy (3.3) on these extra quantum numbers appears only in the order $O(1/N_c)$. Therefore Eq. (3.3) remains valid in the corrected picture of the $O(N_c^0)$ excitations.

The set of the additional quantum numbers responsible for the $O(1/N_c)$ “fine structure” of states depends on the number of quark flavors N_f . Below, we concentrate on the case $N_f = 2$ when the additional quantum system corresponds to an $SU(2)$ rotator. The states of this rotator are described by wave functions $\psi(U)$ depending on the $SU(2)$ matrix U :

$$UU^+ = 1, \quad (3.5)$$

$$\det U = 1. \quad (3.6)$$

The full Hilbert space \mathcal{H} describing the $O(N_c^0)$ excitations is the tensor product of the space \mathcal{H}_U of the rotator wave functions $\psi(U)$ and of the Fock space \mathcal{H}_a of states (3.4) associated with operators a_β^+

$$\mathcal{H} = \mathcal{H}_U \otimes \mathcal{H}_a. \quad (3.7)$$

The states of \mathcal{H} can be considered as linear combinations of the states (3.4) with U dependent coefficients:

$$\psi(U) \prod_{\beta} \left(a_{\beta}^{+}\right)^{n_{\beta}} |B_0\rangle. \quad (3.8)$$

Operators a_β^+ obey the standard commutation relations

$$[a_{\beta}, a_{\beta'}^{+}] = \delta_{\beta\beta'}. \quad (3.9)$$

In fact, β is a multiindex made of several quantum numbers which contain the full information about the corresponding excitation. In the case of baryon-meson scattering states, index β also includes continuous parameters. Operators U and a_β^+ act in different spaces of the tensor product (3.7). Therefore these operators commute:

$$[a_{\beta}, U] = [a_{\beta}^{+}, U] = 0. \quad (3.10)$$

The energy spectrum (3.3) is described by the effective Hamiltonian

$$H = d_1 N_c + d_0 + \sum_{\beta} \Omega_{\beta} a_{\beta}^{+} a_{\beta} + O(N_c^{-1}). \quad (3.11)$$

C. Usual rotational and isospin symmetries

The effective description of the $O(N_c^0)$ baryon excitations and baryon-meson scattering states is constructed for the center of mass frame. In this frame we have the freedom of the 3-dimensional space rotations. This means that in the Hilbert space \mathcal{H} (3.7) we must have a representation of the standard angular momentum algebra (1.8) for the generators J_k corresponding to the total angular momentum of the baryon-meson state (including the orbital

momentum and spin). We also assume the $SU(2)$ flavor symmetry. Its generators T_a obey commutation relations (1.9) and (1.10) with

$$f_{abc} = \varepsilon_{abc} . \quad (3.12)$$

The commutators of the generators J_k, T_a with U are

$$[J_k, U] = \frac{1}{2} U \tau_k , \quad (3.13)$$

$$[T_a, U] = \frac{1}{2} \tau_a^{\text{tr}} U . \quad (3.14)$$

Here τ_a are the Pauli matrices and tr stands for the matrix transposition. In the case of finite transformations, we have

$$\exp(i\omega_k J_k) U \exp(-i\omega_k J_k) = U \exp(i\omega_k \tau_k / 2) , \quad (3.15)$$

$$\exp(i\omega_a T_a) U \exp(-i\omega_a T_a) = \exp(i\omega_a \tau_a^{\text{tr}} / 2) U . \quad (3.16)$$

Now let us turn to the action of generators J_k, T_a on operators a_β^+ . The commutator algebra (3.9) and the form of the Hamiltonian (3.11) do not fix operators a_β^+ completely, since the excitation energies Ω_β are degenerate. Using this freedom, one can choose the operators a_β^+ so that they commute with T_b

$$[T_b, a_\beta^+] = 0 . \quad (3.17)$$

Operators a_β^+ transform under the spatial J_k rotations according to various irreducible representations of $SU(2)$. Therefore, the subscript β can be written as a multiindex

$$\beta = (m, K, K_3) , \quad (3.18)$$

$$a_\beta^+ = a_{mKK_3}^+ . \quad (3.19)$$

We use the special notation K, K_3 for the angular momentum indices, and m stands for the remaining quantum numbers. Under the J_k rotations, we have

$$\exp(i\omega_k J_k) a_{mKK_3}^+ \exp(-i\omega_k J_k) = \sum_{K'_3} a_{mKK'_3}^+ D_{K'_3 K_3}^K (\exp(i\omega_k \tau_k / 2)) . \quad (3.20)$$

Here $D_{K'_3 K_3}^K$ is the standard Wigner function corresponding to the K representation of the group $SU(2)$ (see Appendix A).

According to Eq. (3.17) operators $a_{mKK_3}^+$ are invariant under isospin rotations:

$$\exp(i\omega_b T_b) a_{mKK_3}^+ \exp(-i\omega_b T_b) = a_{mKK_3}^+ . \quad (3.21)$$

In terms of the new notation $a_{mKK_3}^+$ (3.19), relations (3.9) and (3.10) take the form

$$[a_{mKK_3}, a_{m'K'K'_3}^+] = \delta_{mm'} \delta_{KK'} \delta_{K_3 K'_3} , \quad (3.22)$$

$$[U, a_{mKK_3}^+] = [U, a_{mKK_3}] = 0 . \quad (3.23)$$

The states (3.8) can be rewritten in the form

$$\psi(U) \prod_{j=1}^n a_{m^{(j)} K^{(j)} K_3^{(j)}}^+ |B_0\rangle . \quad (3.24)$$

D. Large- N_c contracted $SU(2N_f)$ spin-flavor symmetry

In the previous section, we have demonstrated how the usual rotational symmetry $SO(3) \sim SU(2)$ and the flavor symmetry $SU(N_f)$ (for $N_f = 2$ quark flavors) are realized in the space of baryon-meson scattering states. Now we want to show that this $SU(2)_{\text{spin}} \otimes SU(N_f)_{\text{flavor}}$ symmetry can be extended to the large- N_c contracted spin-flavor $SU(2N_f)$ symmetry. This extension requires the introduction of new generators X_{ia} .

Let us first introduce the $SU(2)$ matrix

$$V = i\tau_2 U. \quad (3.25)$$

Next we define

$$X_{ia} = \frac{1}{2} \text{Sp}(\tau_a V \tau_i V^{-1}) = \frac{1}{2} \text{Sp}(\tau_2 \tau_a \tau_2 U \tau_i U^{-1}). \quad (3.26)$$

Since matrix V is unitary, the X_{ia} is an $SO(3)$ matrix. Therefore

$$X_{ia} X_{ib} = \delta_{ab}, \quad (3.27)$$

$$X_{ia} X_{ja} = \delta_{ij}, \quad (3.28)$$

$$\varepsilon_{ijk} X_{ia} X_{jb} = \varepsilon_{abc} X_{kc}, \quad (3.29)$$

$$\varepsilon_{abc} X_{ia} X_{jb} = \varepsilon_{ijk} X_{kc}. \quad (3.30)$$

Using the definition (3.26) of X_{ia} and the commutators (3.13) and (3.14), one can easily check that operator X_{ia} obeys the commutator algebra (1.11)–(1.13).

Thus, our operators J_i , T_a , X_{ia} obey the full set of the Lie commutators (1.8)–(1.13) of the contracted $SU(2N_f)$ symmetry with $N_f = 2$.

E. Operator \mathcal{K}_i

Let us define the operators

$$\mathcal{K}_i = J_i + X_{ia} T_a, \quad (3.31)$$

$$\mathcal{K}^2 = \mathcal{K}_i \mathcal{K}_i. \quad (3.32)$$

Using relations (1.8)–(1.13), (3.27)–(3.30), one can easily compute the commutators

$$[J_k, \mathcal{K}_l] = i\varepsilon_{klm} \mathcal{K}_m, \quad (3.33)$$

$$[T_a, \mathcal{K}_i] = 0, \quad (3.34)$$

$$[J_i, \mathcal{K}^2] = 0, \quad (3.35)$$

$$[T_a, \mathcal{K}^2] = 0, \quad (3.36)$$

$$[\mathcal{K}_i, X_{jb}] = 0, \quad (3.37)$$

$$[\mathcal{K}_k, \mathcal{K}_l] = i\varepsilon_{klm}\mathcal{K}_m. \quad (3.38)$$

We stress that the commutation relations (3.33)–(3.38) follow directly from the algebra (1.8)–(1.13), (3.27)–(3.30) of the operators J_i, T_a, X_{ia} and from the definition (3.31) of \mathcal{K}_i . In Sec. IV we will find a representation of the algebra (1.11)–(1.13), (3.27)–(3.30) in the space of functionals $A_B(g)$ describing the distribution amplitudes of baryon and baryon-meson states. As a consequence, the commutation relations (3.33)–(3.38) will hold automatically for operators \mathcal{K}_i acting in the space of functionals $A_B(g)$.

Using Eqs. (3.13), (3.14), (3.25), and (3.26), we can also derive

$$[\mathcal{K}_i, V] = 0, \quad (3.39)$$

$$[\mathcal{K}_i, U] = 0. \quad (3.40)$$

The commutators with operators $a_{mKK_3}^+$ can be computed using Eqs. (3.20), (3.21), and (3.23):

$$[X_{jb}, a_{mKK_3}^+] = 0, \quad (3.41)$$

$$[\mathcal{K}_i, a_{mKK_3}^+] = [J_i, a_{mKK_3}^+], \quad (3.42)$$

$$\exp(i\omega_n \mathcal{K}_n) a_{mKK_3}^+ \exp(-i\omega_n \mathcal{K}_n) = \sum_{K'_3} a_{mKK'_3}^+ D_{K'_3 K_3}^K (\exp(i\omega_n \tau_n/2)). \quad (3.43)$$

F. Eigenstates of $J^2, J_3, T^2, T_3, \mathcal{K}^2$

Relations (1.10), (3.35), and (3.36) show that we have the following set of *commuting* operators

$$J^2, J_3, T^2, T_3, \mathcal{K}^2. \quad (3.44)$$

Here we use the short notation (3.32) and

$$J^2 = J_i J_i, \quad T^2 = T_a T_a. \quad (3.45)$$

It is easy to construct the eigenstates of operators (3.44). Let us first consider the state

$$|KK_3\{K^{(j)}m^{(j)}\}\rangle = \left[\sum_{K_3^{(1)} K_3^{(2)} \dots K_3^{(n)}} \mathcal{C}_{K^{(1)}K_3^{(1)}; K^{(2)}K_3^{(2)}; \dots; K^{(n)}K_3^{(n)}}^{KK_3} \prod_{j=1}^n a_{m^{(j)}K^{(j)}K_3^{(j)}}^+ \right] |B_0\rangle, \quad (3.46)$$

where the coefficient $\mathcal{C}_{K^{(1)}K_3^{(1)}; K^{(2)}K_3^{(2)}; \dots; K^{(n)}K_3^{(n)}}^{KK_3}$ is a “multiparticle Clebsh-Gordan coefficient” adding n angular momenta $K^{(1)}K_3^{(1)}, K^{(2)}K_3^{(2)}, \dots, K^{(n)}K_3^{(n)}$ and producing KK_3 . We omit the quantum numbers fixing the details of this addition procedure.

The baryon “ground-state” $|B_0\rangle$ obeys the condition

$$J_i |B_0\rangle = T_a |B_0\rangle = 0. \quad (3.47)$$

Strictly speaking, the state $|B_0\rangle$ with $J = T = 0$ exists only for even N_c . However, formally we can use $|B_0\rangle$ for the construction of other states also in the case of odd N_c , keeping in mind the selection rules for the physically allowed values of J and T .

Using Eqs. (3.31) and (3.47), we find

$$\mathcal{K}_i |B_0\rangle = 0. \quad (3.48)$$

Therefore, we have according to Eq. (3.43)

$$\mathcal{K}_3 |KK_3\{K^{(j)}m^{(j)}\}\rangle = K_3 |KK_3\{K^{(j)}m^{(j)}\}\rangle, \quad (3.49)$$

$$\mathcal{K}^2 |KK_3\{K^{(j)}m^{(j)}\}\rangle = K(K+1) |KK_3\{K^{(j)}m^{(j)}\}\rangle. \quad (3.50)$$

Using Eqs. (3.20) and (3.47), we obtain

$$J_3 |KK_3\{K^{(j)}m^{(j)}\}\rangle = K_3 |KK_3\{K^{(j)}m^{(j)}\}\rangle, \quad (3.51)$$

$$J^2 |KK_3\{K^{(j)}m^{(j)}\}\rangle = K(K+1) |KK_3\{K^{(j)}m^{(j)}\}\rangle. \quad (3.52)$$

Taking into account Eqs. (3.17) and (3.47), we find

$$T_a |KK_3\{K^{(j)}m^{(j)}\}\rangle = 0. \quad (3.53)$$

Now we construct the state

$$|TT_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle = \sum_{T'_3K_3} C_{TT'_3KK_3}^{JJ_3} \sqrt{2T+1} D_{T_3T'_3}^T(U) |KK_3\{K^{(j)}m^{(j)}\}\rangle, \quad (3.54)$$

where $C_{TT'_3KK_3}^{JJ_3}$ is the standard Clebsh-Gordan coefficient.

We obtain from Eqs. (3.15) and (3.16):

$$\exp(i\omega_n J_n) D_{K_3K'_3}^K(U) \exp(-i\omega_n J_n) = D_{K_3K'_3}^K(U \exp(i\omega_n \tau_n/2)), \quad (3.55)$$

$$\exp(i\omega_a T_a) D_{K_3K'_3}^K(U) \exp(-i\omega_a T_a) = D_{K_3K'_3}^K(\exp(i\omega_a \tau_a^{\text{tr}}/2) U). \quad (3.56)$$

Using the properties (3.49)–(3.53) of the state $|KK_3\{K^{(j)}m^{(j)}\}\rangle$ and the transformation rules (3.55), (3.56), we immediately conclude that the state (3.54) is a common eigenstate of the commuting operators (3.44).

Indeed, the state $|KK_3\{K^{(j)}m^{(j)}\}\rangle$ has zero isospin according to Eq. (3.53). Therefore, the isospin T, T_3 of the state (3.54) is generated by $D_{T_3T'_3}^T(U)$.

Next, the factor $D_{T_3T'_3}^T(U)$ has zero \mathcal{K}_i momentum according to Eq. (3.40). Therefore, the quantum number \mathcal{K} of the state (3.54) is inherited from $|KK_3\{K^{(j)}m^{(j)}\}\rangle$.

As for the angular momentum J, J_3 of the state (3.54), it comes from the angular momentum T, T'_3 of $D_{T_3T'_3}^T(U)$ and from the angular momentum K, K_3 of the state $|KK_3\{K^{(j)}m^{(j)}\}\rangle$. The Clebsh-Gordan coefficient $C_{TT'_3KK_3}^{JJ_3}$ adds these two contributions producing the total angular momentum J, J_3 of the state (3.54).

Thus, under spatial rotations and isospin transformations of the state (3.54) we have

$$\exp(i\omega_n J_n) |TT_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle = \sum_{J'_3} |TT_3JJ'_3K\{K^{(j)}m^{(j)}\}\rangle D_{J'_3J_3}^J(\exp(i\omega_n \tau_n/2)), \quad (3.57)$$

$$\exp(i\omega_a T_a) |TT_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle = \sum_{T'_3} |TT'_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle D_{T'_3T_3}^T(\exp(i\omega_a \tau_a/2)). \quad (3.58)$$

Inserting Eq. (3.46) into Eq. (3.54), we obtain

$$\begin{aligned} |TT_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle &= \sum_{T'_3K_3} C_{TT'_3KK_3}^{JJ_3} \sqrt{2T+1} D_{T_3T'_3}^T(U) \\ &\times \left[\sum_{K_3^{(1)}K_3^{(2)}\dots K_3^{(n)}} \mathcal{C}_{K^{(1)}K_3^{(1)};K^{(2)}K_3^{(2)};\dots;K^{(n)}K_3^{(n)}}^{KK_3} \prod_{j=1}^n a_{m^{(j)}K^{(j)}K_3^{(j)}}^+ \right] |B_0\rangle. \end{aligned} \quad (3.59)$$

The excitation energies Ω_β (3.3) associated with operators $a_\beta^+ = a_{mKK_3}^+$ (3.19) depend only on m and K but not on K_3 . Therefore, the energy of the state (3.59) is

$$E = N_c d_1 + d_0 + \sum_{j=1}^n \Omega_{m^{(j)}K^{(j)}} + O(1/N_c). \quad (3.60)$$

Using the standard expression for the Clebsh-Gordan coefficients via the $3j$ symbols [see Eq. (A14) in Appendix A], we can rewrite Eq. (3.59) in the form

$$\begin{aligned} |TT_3JJ_3K\{K^{(j)}m^{(j)}\}\rangle &= \sqrt{(2T+1)(2J+1)} \sum_{T'_3K_3} D_{T'_3T_3}^T(U) (-1)^{T-K+J_3} \\ &\times \begin{pmatrix} T & K & J \\ T'_3 & K_3 & -J_3 \end{pmatrix} \left[\sum_{K_3^{(1)}K_3^{(2)}\dots K_3^{(n)}} \mathcal{C}_{K^{(1)}K_3^{(1)};K^{(2)}K_3^{(2)};\dots;K^{(n)}K_3^{(n)}}^{KK_3} \prod_{j=1}^n a_{m^{(j)}K^{(j)}K_3^{(j)}}^+ \right] |B_0\rangle. \end{aligned} \quad (3.61)$$

G. Simplest states

The lowest baryon states correspond to $n = 0$ in Eq. (3.60), i.e. to the absence of the $a_{m^{(j)}K^{(j)}K_3^{(j)}}^+$ excitations in Eq. (3.59). In this case we have $K = K_3 = 0$ in Eq. (3.59) and the Clebsh-Gordan coefficient $C_{TT'_3KK_3}^{JJ_3}$ leads to the constraint

$$J = T. \quad (3.62)$$

As a result, one arrives at the states

$$|T = J, T_3J_3\rangle = \sqrt{2T+1} D_{T_3T_3}^{T=J}(U). \quad (3.63)$$

The masses of these states are given by Eq. (3.2).

In the case of single $a_{m^{(j)}K^{(j)}K_3^{(j)}}^+$ excitations corresponding to $n = 1$ in Eq. (3.59), we arrive at the states of the form

$$|TT_3JJ_3Km\rangle = \sum_{T'_3K_3} C_{TT'_3KK_3}^{JJ_3} \sqrt{2T+1} D_{T'_3T_3}^T(U) a_{mKK_3}^+ |B_0\rangle. \quad (3.64)$$

IV. FACTORIZATION PROPERTIES OF FUNCTIONALS $A_B(g)$

A. Functionals $A_B(g)$ as a representation for baryon-meson states

In contrast to the universal functional $W(g)$, the functional $A_B(g)$ appearing in Eq. (2.15) depends on the baryon-meson state $|B\rangle$. In some sense, the functionals $A_B(g)$ can be considered as a special representation for the baryon-meson states $|B\rangle$. Strictly speaking, this point of view is not quite correct because of at least two reasons.

First, we cannot guarantee that any different baryon-meson states $|B_1\rangle$ and $|B_2\rangle$ generate different functionals $A_{B_1}(g)$ and $A_{B_2}(g)$. Second, the definition (2.15) of the functional $A_B(g)$ assumes the separation of the power factor $N_c^{\nu_B}$. Generally speaking, the power ν_B depends on the state $|B\rangle$.

Nevertheless the interpretation of the functionals $A_B(g)$ in terms of a new representation for baryon-meson states is rather helpful for the understanding of the formal relations which are discussed below. We will often use this interpretation below (keeping in mind its limitations).

The algebra of operators a_{mKK_3} , $a_{mKK_3}^+$, and U described in the previous section must have some representation in the space of functionals $A_B(g)$. According to Eq. (3.23), operators U and $a_{mKK_3}^+$ commute. This means that there must exist a representation in which U and $a_{mKK_3}^+$ are diagonal. Our conjecture is that the representation for baryon-meson states $|B\rangle$ in terms of functionals $A_B(g)$ gives this kind of representation:

$$U_{fs} A_B(g) = Q_{fs}(g) A_B(g), \quad (4.1)$$

$$a_{mKK_3}^+ A_B(g) = \xi_{mKK_3}(g) A_B(g), \quad (4.2)$$

where $Q_{fs}(g)$ and $\xi_{mKK_3}(g)$ are some functionals.

B. Properties of functional $Q_{fs}(g)$

According to Eq. (3.6), the matrix U has a unit determinant. Therefore relation (4.1) immediately leads us to the conclusion that

$$\det_{fs} Q_{fs}(g) = 1. \quad (4.3)$$

Although the matrix $Q_{fs}(g)$ belongs to $SL(2, C)$, it is not unitary. Indeed, the functionals $A_B(g)$ are analytical in g . Representation (4.1), (4.2) must be compatible with the analyticity of $A_B(g)$. This means that functionals $Q_{fs}(g)$ and $\xi_{mKK_3}(g)$ are also analytical in g . The analyticity of $Q_{fs}(g)$ excludes its unitarity.

Below we will often work with Wigner functions $D_{mm'}^J(R)$ extended to the $SL(2, C)$ matrices R . As is well known, the finite-dimensional irreducible representations of $SL(2, C)$ are parametrized by two spins (J_1, J_2) . Let us concentrate on the representations with $J_2 = 0$. One can consider these representations as an analytical continuation of the standard Wigner functions $D_{mm'}^{J_1}(R)$ from $SU(2)$ to $SL(2, C)$. Below we use notation $D_{mm'}^{J_1}(R)$ for $SL(2, C)$ matrices R assuming that $J_2 = 0$. It is also well known that Wigner functions $D_{mm'}^J(R)$ can be represented as polynomials in matrix elements of R if R belongs to $SL(2, C)$. The inverse is also true: Any polynomial $P(R)$ of matrix elements of R can be decomposed into a linear combination of functions $D_{mm'}^J(R)$:

$$P(R) = \sum_{Jmm'} c_{Jmm'} D_{mm'}^J(R). \quad (4.4)$$

Since $Q(g)$ belongs to $SL(2, C)$, we can take $R = Q(g)$ in this decomposition:

$$P(Q(g)) = \sum_{Jmm'} c_{Jmm'} D_{mm'}^J(Q(g)). \quad (4.5)$$

We see that functions $D_{mm'}^J(Q(g))$ make a basis in the space of polynomials $P(Q(g))$.

C. Functional $\Xi_B(g)$

In the large- N_c world with even N_c we can take the ground state baryon B_0 (3.47) with $J = T = 0$ and denote its functional $A_0(g, t)$. Now let us define

$$\Xi_B(g) = \frac{A_B(g)}{A_0(g)}. \quad (4.6)$$

This definition can be used for any N_c (including odd values), although functional $A_0(g)$ is taken from the even- N_c case. The functional $A_0(g)$ corresponds to the state $|B_0\rangle$ appearing in Eq. (3.61). Note that the baryon state with $J = T = 1/2$ is possible only for odd N_c whereas the functional $A_0(g)$, used in the definition of $\Xi_B(g)$ (4.6), was defined in the large- N_c world with even N_c . This mixed odd-even N_c construction appears because $Q(g)$ describes elementary excitations of the *fermion* type.

If we accept relations (4.1) and (4.2), then the calculation of functionals $\Xi_{TT_3JJ_3K\{K^{(j)}m^{(j)}\}}(g)$ for the $O(N_c^{-1})$ and $O(N_c^0)$ excited baryon and baryon-meson states (3.61) becomes trivial: the state (3.61) is mapped to the functional

$$\begin{aligned} \Xi_{TT_3JJ_3K\{K^{(j)}m^{(j)}\}}(g) &= \sqrt{(2T+1)(2J+1)} \sum_{T'_3K_3} D_{T_3T'_3}^T(Q(g)) (-1)^{T-K+J_3} \\ &\times \begin{pmatrix} T & K & J \\ T'_3 & K_3 & -J_3 \end{pmatrix} \sum_{K_3^{(1)}K_3^{(2)}\dots K_3^{(n)}} c_{K^{(1)}K_3^{(1)};K^{(2)}K_3^{(2)};\dots;K^{(n)}K_3^{(n)}}^{KK_3} \prod_{j=1}^n \xi_{m^{(j)}K^{(j)}K_3^{(j)}}(g). \end{aligned} \quad (4.7)$$

The Wigner function $D_{T_3 T'_3}^T(Q(g))$ should be understood in the sense of the $SL(2, C)$ complexification of the $SU(2)$ representations, as was explained in Sec. IV B.

In the case of single excitations ($n = 1$) we find from Eq. (4.7)

$$\begin{aligned} \Xi_{TT_3 JJ_3 K m}(g) &= \sqrt{(2T+1)(2J+1)} \sum_{T'_3 K_3} D_{T_3 T'_3}^T(Q(g)) (-1)^{T-K+J_3} \\ &\times \begin{pmatrix} T & K & J \\ T'_3 & K_3 & -J_3 \end{pmatrix} \xi_{m K K_3}(g). \end{aligned} \quad (4.8)$$

The lowest $O(N_c^{-1})$ excitations (3.63) correspond to $n = 0$ in Eq. (4.7):

$$\Xi_{T=J, T_3 J_3}(g) = \sqrt{2T+1} D_{T_3 J_3}^T(Q(g)). \quad (4.9)$$

D. Consistency with the evolution equation

Let us check the consistency of the representation (4.7) with the evolution equation (2.23). Indeed, taking the difference of equations (2.23) for an excited state A_B and for A_0 , we find

$$\frac{\partial \Xi_B(g, t)}{\partial t} = - \left\{ (g \otimes g) \cdot \bar{K} \cdot \left[\frac{\delta W(g, t)}{\delta g} \otimes \frac{\delta \Xi_B(g, t)}{\delta g} \right] \right\}. \quad (4.10)$$

Functionals $\Xi_B(g, t)$ (4.6) are associated with baryon (baryon-meson) states $|B\rangle$. As explained in Sec. IV A, one can think about functionals $\Xi_B(g, t)$ as a special representation for $O(N_c^{-1})$ and $O(N_c^0)$ excited states in large- N_c QCD. In particular, we have an implementation of QCD symmetries in the space of functionals $\Xi_B(g, t)$.

Solutions of Eq. (4.10) have an important property. If $\{\Xi_n(g, t)\}$ is some set of solutions, then any function of these solutions

$$F(\Xi_1(g, t), \Xi_2(g, t), \dots) \quad (4.11)$$

is also a solution of the same equation:

$$\frac{\partial F}{\partial t} = - \left[(g \otimes g) \cdot \bar{K} \cdot \left(\frac{\delta W}{\delta g} \otimes \frac{\delta F}{\delta g} \right) \right]. \quad (4.12)$$

This property of the solutions of the evolution equation (4.10) leads us to the idea that there must exist a set of “elementary” functionals $\Xi_n(g, t)$ such that an arbitrary solution $\Xi_B(g, t)$ corresponding to some baryon-meson state can be represented as a functional of the elementary solutions $\Xi_n(g, t)$. Note that the decomposition (4.7) gives exactly this type of representation. Indeed, according to Eq. (4.7) the functional $\Xi_{TT_3 JJ_3 K \{K^{(j)} m^{(j)}\}}(g)$ is a linear combination of functionals

$$D_{T_3 T'_3}^T[Q(g, t)] \prod_{j=1}^n \xi_{m^{(j)} K^{(j)} K_3^{(j)}}(g, t). \quad (4.13)$$

Here we have explicitly marked the dependence of the functionals $Q(g, t)$ and $\xi_{m K K_3}(g, t)$ on the scale t . Note that $D_{T_3 T'_3}^T[Q(g, t)]$ can be represented as a polynomial in matrix elements of $Q(g, t)$. Therefore $\Xi_{TT_3 JJ_3 K \{K^{(j)} m^{(j)}\}}(g)$ (4.7) is a polynomial in “elementary” functionals $Q(g, t)$ and $\xi_{m K K_3}(g, t)$. If these elementary functionals obey Eq. (4.10), then the “composite” functional $\Xi_{TT_3 JJ_3 K \{K^{(j)} m^{(j)}\}}(g)$ automatically satisfies the same Eq. (4.10) according to the property (4.12). The same argument shows that Eq. (4.12) also holds for $F = \det Q(g, t)$. This means that the evolution equation is consistent with the constraint (4.3).

Let us summarize. In this section we have explicitly checked the consistency of Eqs. (4.10) and (4.7). The evolution equation (4.10) comes from perturbative QCD, whereas Eq. (4.7) describes the non-perturbative properties of baryons based on the contracted spin-flavor group. Thus the two different aspects of hadronic physics perfectly match in our large- N_c equations.

E. Symmetries

1. Transformations of g

In Sec. IV A we described the action of the operators U and $a_{mKK_3}^+$ on functionals $A_B(g)$. Now we want to study the action of operators J_i and T_a . The realization of the isospin symmetry in the space of functionals $A_B(g)$ is straightforward. Starting from the relation

$$[T_a, g_{fs}] = \frac{1}{2}(\tau_a)_{f'f} g_{f's}, \quad (4.14)$$

we obtain

$$\exp(i\omega_a T_a) g \exp(-i\omega_a T_a) = \exp(i\omega_a \tau_a^{\text{tr}}/2) g \quad (4.15)$$

so that for any functional $F(g)$ we have

$$[\exp(i\omega_a T_a) F \exp(-i\omega_a T_a)](g) = F(\exp(i\omega_a \tau_a^{\text{tr}}/2) g). \quad (4.16)$$

With the generators of space rotations J_i the situation is more subtle. In the light-cone formalism the rotational symmetry is “broken” by the light-cone vector n appearing in the definition of the baryon distribution amplitude (2.1), (2.5). Actually this symmetry is not broken but simply becomes “hidden”. Therefore, simple transformation rules in terms of g can be written only for the axial rotations around the vector n . Assuming that this light-cone vector is directed along the third axis, we can write the transformation rule for axial rotations, which can be applied to an arbitrary functional $F(g)$:

$$[\exp(i\omega_3 J_3) F \exp(-i\omega_3 J_3)](g) = F(g \exp(i\omega_3 \tau_3/2)). \quad (4.17)$$

2. Transformation rules for $Q(g)$

Although we cannot write a general formula for the action of generators J_1, J_2 on an arbitrary functional $F(g)$, we know how these generators act on some special functionals. For example, comparing Eqs. (3.15) and (3.16) with Eq. (4.1), we conclude that

$$\exp(i\omega_k J_k) Q(g) \exp(-i\omega_k J_k) = Q(g) \exp(i\omega_k \tau_k/2), \quad (4.18)$$

$$\exp(i\omega_a T_a) Q(g) \exp(-i\omega_a T_a) = \exp(i\omega_a \tau_a^{\text{tr}}/2) Q(g). \quad (4.19)$$

In the infinitesimal form we have

$$[J_k, Q_{fs}(g)] = \frac{1}{2} Q_{fs'}(g) (\tau_k)_{s's}, \quad (4.20)$$

$$[T_a, Q_{fs}(g)] = \frac{1}{2} Q_{f's}(g) (\tau_a)_{f'f}. \quad (4.21)$$

Combining Eqs. (4.16) and (4.19), we find

$$Q(\exp(i\omega_k \tau_k/2) g) = \exp(i\omega_k \tau_k/2) Q(g). \quad (4.22)$$

3. Symmetries of ξ

According to Eqs. (3.20), (3.21), and (4.2) we have

$$[T_a, \xi_{mKK_3}(g)] = 0, \quad (4.23)$$

$$\exp(i\omega_j J_j) \xi_{mKK_3}(g) \exp(-i\omega_j J_j) = \sum_{K'_3} \xi_{mKK'_3} D_{K'_3 K_3}^{(K)} (\exp(i\omega_j \tau_j/2)), \quad (4.24)$$

Both $\xi_{mKK_3}(g)$ and $Q(g)$ are “multiplication operators”. Therefore they commute

$$[\xi_{mKK_3}(g), Q_{fs}(g)] = 0. \quad (4.25)$$

F. Spin-flavor symmetry

By analogy with Eq. (3.25), we define

$$\tilde{Q}(g) = i\tau_2 Q(g). \quad (4.26)$$

Then we find from Eqs. (4.20) and (4.21)

$$[J_i, \tilde{Q}_{fs}(g)] = \frac{1}{2} \tilde{Q}_{fs'}(g) (\tau_i)_{s's}, \quad (4.27)$$

$$[T_a, \tilde{Q}_{fs}(g)] = -\frac{1}{2} (\tau_a)_{ff'} \tilde{Q}_{f's}(g). \quad (4.28)$$

For $SL(2, C)$ matrices R , we have the general relation

$$(i\tau_2 R)^{-1} = (R i\tau_2)^{\text{tr}}. \quad (4.29)$$

Applying this identity to U and $\tilde{Q}(g)$, we obtain

$$(i\tau_2 U)^{-1} = (U i\tau_2)^{\text{tr}}, \quad [i\tau_2 Q(g)]^{-1} = [Q(g) i\tau_2]^{\text{tr}}. \quad (4.30)$$

Combining these relations with Eq. (4.1), we find

$$[(i\tau_2 U)^{-1}]_{fs} A_B(g) = \left\{ [i\tau_2 Q(g)]^{-1} \right\}_{fs} A_B(g). \quad (4.31)$$

In terms of matrices V (3.25) and $\tilde{Q}(g)$ (4.26), this relation becomes

$$(V^{-1})_{fs} A_B(g) = \left\{ [\tilde{Q}(g)]^{-1} \right\}_{fs} A_B(g). \quad (4.32)$$

Now we derive from Eqs. (3.26), (4.1), and (4.32)

$$X_{ia} A_B(g) = \frac{1}{2} \text{Sp} \left\{ \tilde{Q}(g) \tau_i [\tilde{Q}(g)]^{-1} \tau_a \right\} A_B(g). \quad (4.33)$$

Thus the action of the operator X_{ia} on $A_B(g)$ reduces to the factor

$$X_{ia} = \frac{1}{2} \text{Sp} \left\{ \tilde{Q}(g) \tau_i [\tilde{Q}(g)]^{-1} \tau_a \right\}. \quad (4.34)$$

Using this expression for X_{ia} and commutation relations (4.27), (4.28), it is straightforward to show that this operator obeys all spin-flavor algebra relations (1.11)–(1.13), (3.27)–(3.30). Note that for the derivation of these relations one does not need the unitarity of $\tilde{Q}(g)$; it is sufficient that $\tilde{Q}(g)$ belongs to $SL(2, C)$.

Thus, we have shown that our conjecture about the realization (4.1), (4.2) of the operators U and $a_{mKK_3}^+$ in the space of functionals $A_B(g)$ is consistent with the algebra of the spin-flavor symmetry (1.11)–(1.13), (3.27)–(3.30).

Now, when we have the set of operators T_a, J_i, X_{ia} acting on functionals $A_B(g)$ and obeying the algebraic relations (1.8)–(1.13), (3.27)–(3.30), we can use Eq. (3.31) in order to define operator \mathcal{K}_i acting on functionals $A_B(g)$. This operator \mathcal{K}_i automatically obeys the commutation relations (3.33)–(3.38), since these commutators follow directly from the algebraic relations (1.8)–(1.13), (3.27)–(3.30), which have been already checked for our representation of the spin-flavor algebra in the space of functionals $A_B(g)$.

Using relations (4.23)–(4.25) and the definition (3.31) of the operator \mathcal{K}_i , we find

$$\exp(i\omega_j \mathcal{K}_j) \xi_{mKK_3}(g) \exp(-i\omega_j \mathcal{K}_j) = \sum_{K'_3} \xi_{mKK'_3} D_{K'_3 K_3}^{(K)} (\exp(i\omega_j \mathcal{K}_j/2)). \quad (4.35)$$

This relation shows that the functional $\xi_{mKK_3}(g)$ belongs to the K representation of the $SU(2)$ group generated by operators \mathcal{K}_i .

Our representation of the spin-flavor algebra in the space of functionals $A_B(g)$ will allow us to work with the set of commuting operators $J^2, J_3, T^2, T_3, \mathcal{K}^2$ (3.44). Obviously the eigenstates of these operators are described by the functionals (4.7).

G. Factors $N_c^{\nu_B}$

The definition (2.15) of the functional $A_B(g)$ assumes the separation of the power factor $N_c^{\nu_B}$. In the case of states $|B\rangle$ belonging to the same irreducible representation of the large- N_c spin-flavor symmetry group we have a common power ν_B for all these states. However, in the general case parameter ν_B depends on the state $|B\rangle$.

Creating an extra *nonzero mode* excitation $\xi_{m(j)K(j)K_3^{(j)}}(g)$ in Eq. (4.7), we shift ν_B by $1/2$

$$\nu_B \rightarrow \nu_B + \frac{1}{2}. \quad (4.36)$$

In order to derive this property, let us introduce the short notation $\mathcal{B}(g)$ for the baryon operator

$$\mathcal{B}(g) = N_c^{-N_c/2} \prod_{c=1}^{N_c} \left[\sum_{fs} \int dy g_{fs}(y) \chi_{cfs} \left(\frac{y}{N_c} \right) \right] \quad (4.37)$$

made of fields $\chi_{cfs}(y)$ (2.5). Using Eqs. (2.1), (2.12), and (2.13), we can rewrite Eq. (2.15) in the form

$$\langle 0 | \mathcal{B}(g) | B \rangle = N_c^{\nu_B} A_B(g) \exp [N_c W(g)] . \quad (4.38)$$

Let J be a color-singlet meson operator

$$J = \sum_{c=1}^{N_c} \bar{\psi}_c \Gamma \psi_c , \quad (4.39)$$

where Γ is some spin-flavor matrix. Then by analogy with Eq. (4.38) we can write

$$\langle 0 | \mathcal{B}(g) J | B \rangle = N_c^{\nu_{BJ}} A_{BJ}(g) \exp [N_c W(g)] . \quad (4.40)$$

Note that the exponential behavior in Eqs. (4.38) and (4.40) is determined by the same universal functional $W(g)$, as was explained in Ref. [11]. However, functional $A_{BJ}(g)$ and parameter ν_{BJ} are different from $A_B(g)$ and ν_B .

Obviously the presence of the operator J in Eq. (4.40) leads to an extra power of N_c . Therefore comparing Eqs. (4.38) and (4.40), we find

$$\nu_{BJ} = \nu_B + 1 . \quad (4.41)$$

The operator J creates an extra meson M in the state $|B\rangle$:

$$J | B \rangle \rightarrow | BM \rangle . \quad (4.42)$$

Now we can write the large- N_c asymptotic expression (2.15) for the state $|BM\rangle$:

$$\langle 0 | \mathcal{B}(g) | BM \rangle = N_c^{\nu_{BM}} A_{BM}(g) \exp [N_c W(g)] . \quad (4.43)$$

At large N_c the connected part of the correlation function of two currents J has the behavior $\langle 0 | JJ | 0 \rangle_{\text{conn.}} \sim N_c$. This allows us to estimate the matrix element $\langle 0 | J | M \rangle \sim \sqrt{N_c}$ corresponding to the intermediate meson state M . Comparing this result with Eqs. (4.40) and (4.43), we conclude that $\nu_{BJ} = \nu_{BM} + 1/2$. Together with Eq. (4.41) this yields

$$\nu_{BM} = \nu_B + \frac{1}{2} . \quad (4.44)$$

This completes the derivation of the rule (4.36).

We stress that the shift of ν_B (4.36) is associated only with the nonzero mode factors $\xi_{m(j)K(j)K_3^{(j)}}(g)$ in Eq. (4.7). The zero mode factor $D_{T_3 T_3'}^T [Q(g)]$ does not change ν_B since this factor is associated with the transformations of the large- N_c spin-flavor symmetry group. These transformations connect different baryons belonging to the same representation of the spin-flavor symmetry group. Parameter ν_B should be the same for all baryons within this multiplet.

H. Lowest baryons and N - Δ relation

The lowest $O(N_c^{-1})$ excited baryons are described by functionals $\Xi_{T=J, T_3 J_3}(g)$ (4.9). Combining Eqs. (4.6) and (4.9), we obtain

$$A_{T_3 J_3}^{T=J}(g) = \sqrt{2T+1} D_{T_3 J_3}^T [Q(g)] A_0(g). \quad (4.45)$$

As was explained in Sec. IV B, we use the short notation D^T implying the $SL(2, C)$ Wigner function $D^{(T, \dot{0})}$. The $SL(2, C)$ Wigner function $D^{(J_1, J_2)}(R)$ with $J_1 = 1/2$, $J_2 = 0$ is

$$D_{mm'}^{(1/2, \dot{0})}(R) = R_{mm'}. \quad (4.46)$$

Therefore, we obtain from Eq. (4.45) for the nucleon case ($J = T = 1/2$):

$$A_{T_3 J_3}^{(N)}(g) = \sqrt{2} Q_{T_3 J_3}(g) A_0(g). \quad (4.47)$$

Taking into account Eq. (4.3), we find

$$A_0(g) = \sqrt{\frac{1}{2} \det_{T_3 J_3} A_{T_3 J_3}^{(N)}(g)}, \quad (4.48)$$

$$Q_{T_3 J_3}(g) = \frac{A_{T_3 J_3}^{(N)}(g)}{\sqrt{\det A^{(N)}(g)}}. \quad (4.49)$$

Inserting these expressions into Eq. (4.45), we obtain

$$A_{T_3 J_3}^{J=T}(g) = D_{T_3 J_3}^T \left(\frac{A^{(N)}(g)}{\sqrt{\det A^{(N)}(g)}} \right) \sqrt{\left(T + \frac{1}{2}\right) \det A^{(N)}(g)}. \quad (4.50)$$

In particular, we find for the Δ resonance ($J = T = 3/2$):

$$A_{T_3 J_3}^{(\Delta)}(g) = D_{T_3 J_3}^{3/2} \left(\frac{A^{(N)}(g)}{\sqrt{\det A^{(N)}(g)}} \right) \sqrt{2 \det A^{(N)}(g)}. \quad (4.51)$$

Now let us consider the functionals $\Phi_{T_3 J_3}^{(N)}(g)$ and $\Phi_{T_3 J_3}^{(\Delta)}(g)$ (2.13) describing the distribution amplitude of the nucleon and the Δ . The large- N_c behavior of these functionals is given by Eq. (2.15). We have $\nu_\Delta = \nu_N$ in Eq. (2.15) since the nucleon and the Δ resonance belong to the same irreducible representation of the spin-flavor symmetry. Combining Eqs. (2.15) and (4.51), we find

$$\Phi_{T_3 J_3}^{(\Delta)}(g) = D_{T_3 J_3}^{3/2} \left(\frac{\Phi^{(N)}(g)}{\sqrt{\det \Phi^{(N)}(g)}} \right) \sqrt{2 \det \Phi^{(N)}(g)} [1 + O(N_c^{-1})]. \quad (4.52)$$

One should not try to use this relation for the derivation of a direct expression for the distribution amplitude $\Psi^{(\Delta)}(x_1, x_2, x_3)$ of the Δ resonance via the nucleon distribution amplitudes $\Psi^{(N)}(x_1, x_2, x_3)$ at finite $N_c = 3$. Although at finite N_c we have the exact expression (2.14) for the distribution amplitude $\Psi^{(\Delta)}$ ($\Psi^{(N)}$) via the functional $\Phi^{(\Delta)}(g)$ [$\Phi^{(N)}(g)$], one cannot combine this exact relation with the large- N_c expression (4.52), because the number of variational derivatives in Eq. (2.14) grows as N_c .

Therefore practical applications of Eq. (4.52) should deal directly with the functionals $\Phi^{(N)}(g)$, $\Phi^{(\Delta)}(g)$ which can be considered as g moments of the corresponding distribution amplitudes.

V. SOFT-PION THEOREM

A. Soft-pion theorem at finite N_c

The soft-pion theorem for the distribution amplitude of the near-threshold nucleon-pion state was derived and successfully applied to the description of SLAC data on the near-threshold pion production in deeply virtual scattering

in Ref. [33]. We want to derive the large- N_c version of this theorem and to use it for the consistency check of the representation (4.6), (4.7) for the functional $A_B(g)$.

We start from the general soft-pion theorem valid for any operator \mathcal{O} at finite N_c ,

$$\begin{aligned} \langle 0 | \mathcal{O} | N(T_3 J_3), \pi_a(\mathbf{k}) \rangle &= -\frac{i}{F_\pi} \langle 0 | [Q_{5a}, \mathcal{O}] | N(T_3 J_3) \rangle \\ &+ \frac{ig_A}{2F_\pi} \sum_{T'_3 J'_3 n} (\tau_a)_{T'_3 T_3} (\tau_n)_{J'_3 J_3} \frac{k^n}{E_\pi} \langle 0 | \mathcal{O} | N(T'_3 J'_3) \rangle. \end{aligned} \quad (5.1)$$

Here $|N(T_3 J_3), \pi_a(\mathbf{k})\rangle$ is a state containing

- nucleon $N(T_3 J_3)$ in the rest frame with spin J_3 and isospin T_3 ,
- soft pion $\pi_a(\mathbf{k})$ with small momentum \mathbf{k} , energy $E_\pi = \sqrt{m_\pi^2 + |\mathbf{k}|^2}$ and flavor index $a = 1, 2, 3$.

Note that for soft pions the difference between the *in* and *out* scattering states can be ignored. The difference between the nucleon rest frame and the c.m. frame is also not important in the soft-pion case.

The first term on the RHS of Eq. (5.1) contains the commutator of the operator \mathcal{O} with the axial charge

$$Q_{5a} = \int d^3x \bar{q} \frac{\tau_a}{2} \gamma_0 \gamma_5 q, \quad (5.2)$$

$$[Q_{5a}, q] = -\frac{1}{2} \gamma_5 \tau_a q. \quad (5.3)$$

The commutator term in Eq. (5.1) describes the S wave of the pion-nucleon state. This term does not vanish in the threshold limit $\mathbf{k} \rightarrow 0$. It is accompanied by $1/F_\pi$, where

$$F_\pi = 93 \text{ MeV} \quad (5.4)$$

is the pion decay constant.

The second term on the RHS of Eq. (5.1) originates from the pion-nucleon vertex and its contribution is proportional to the Goldberger-Treiman ratio

$$\frac{g_A}{F_\pi} = \frac{g_{\pi NN}}{M_N}. \quad (5.5)$$

Here g_A is the nucleon axial constant, M_N is the nucleon mass and $g_{\pi NN}$ is the pion-nucleon constant. The vertex also contains the spin Pauli matrix $(\tau_n)_{J'_3 J_3}$ and the isospin Pauli matrix $(\tau_a)_{T'_3 T_3}$. Obviously the second term on the RHS of Eq. (5.1) describes the P wave. This term vanishes at the threshold $\mathbf{k} = \mathbf{0}$.

B. Soft-pion theorem at large N_c

The soft-pion theorem (5.1) is valid only for those pions whose energy E_π is much smaller than the Δ production threshold:

$$E_\pi \ll M_\Delta - M_N. \quad (5.6)$$

In the limit $N_c \rightarrow \infty$ we have

$$M_\Delta - M_N = O(N_c^{-1}). \quad (5.7)$$

If we combine the condition (5.6) with the large- N_c limit, then the pion energy E_π should approach zero much faster than $1/N_c$.

In this section we want to study *another* regime when the large- N_c limit is taken *before* the soft-pion limit:

$$M_\Delta - M_N = O(N_c^{-1}) \ll E_\pi \ll \Lambda_{\text{QCD}}. \quad (5.8)$$

In the real world, where $m_\pi < M_\Delta - M_N$, this regime is certainly unphysical. However, our current aim is not the practical applications of the soft-pion theorem. We want to use the large- N_c version of the soft-pion theorem as a theoretical check of the consistency of the factorized representation for the functional $A_B(g)$ (4.6), (4.8).

In the region (5.8) we must modify the soft-pion theorem (5.1) replacing the last term on the RHS by the sum over all $O(N_c^{-1})$ excited baryons which can appear in the intermediate state

$$\begin{aligned} \langle 0 | \mathcal{O} | B(T = J, T_3 J_3), \pi_a(\mathbf{k}) \rangle &= -\frac{i}{F_\pi} \langle 0 | [Q_{5a}, \mathcal{O}] | B(T = J, T_3 J_3) \rangle \\ &+ \sum_{T' T'_3 J'_3 n} \frac{i}{2F_\pi} G_A^{T' T'_3 J'_3; T T_3 J_3; an} \frac{k^n}{E_\pi} \langle 0 | \mathcal{O} | B(T' = J', T'_3 J'_3) \rangle. \end{aligned} \quad (5.9)$$

Here $G_A^{T' T'_3 J'_3; T T_3 J_3; an}$ is the matrix element of the axial current corresponding to the transition between the baryons $B(T = J, T_3 J_3)$ and $B(T' = J', T'_3 J'_3)$. For example, in the nucleon case ($T = T' = 1/2$) we have

$$G_A^{1/2, T'_3 J'_3; 1/2, T_3 J_3; an} = g_A(\tau_a)_{T'_3 J'_3}(\tau_n)_{J_3 J_3} \quad (5.10)$$

in agreement with the structure of the second term on the RHS of Eq. (5.1). The general large- N_c expression for $G_A^{T' T'_3 J'_3; T T_3 J_3; an}$ can be found in Appendix B.

Let us take operator (4.37) for \mathcal{O} in the soft-pion theorem (5.9):

$$\mathcal{O} = \mathcal{B}(g). \quad (5.11)$$

According to Eq. (5.3) we have for this operator

$$[Q_{5a}, \mathcal{B}(g)] = - \sum_{f_1 f_2 s_1 s_2} \frac{1}{2} \int dy g_{f_1 s_1}(y) (\tau_a)_{f_1 f_2} (\gamma_5)_{s_1 s_2} \frac{\delta}{\delta g_{f_2 s_2}(y)} \mathcal{B}(g). \quad (5.12)$$

Now we find from Eq. (5.9)

$$\begin{aligned} \langle 0 | \mathcal{B}(g) | B(T = J, T_3 J_3), \pi_a(\mathbf{k}) \rangle &= \frac{i}{2F_\pi} \sum_{f_1 f_2 s_1 s_2} \int dy g_{f_1 s_1}(y) (\tau_a)_{f_1 f_2} (\gamma_5)_{s_1 s_2} \frac{\delta}{\delta g_{f_2 s_2}(y)} \langle 0 | \mathcal{B}(g) | B(T = J, T_3 J_3) \rangle \\ &+ \sum_{T' T'_3 J'_3 n} \frac{i}{2F_\pi} G_A^{T' T'_3 J'_3; T T_3 J_3; an} \frac{k^n}{E_\pi} \langle 0 | \mathcal{B}(g) | B(T' = J', T'_3 J'_3) \rangle. \end{aligned} \quad (5.13)$$

C. Functional $A_{B\pi}(g)$ for pion-baryon states

Using Eq. (5.13), we can compute the functional $A_{B\pi}(g)$ corresponding to the large- N_c distribution amplitude of the pion-baryon scattering state with a soft pion. According to Eqs. (2.15) and (4.44) we expect that at large N_c

$$\langle 0 | \mathcal{B}(g) | B(T = J, T_3 J_3) \rangle = N_c^{\nu_B} A_{B(T=J, T_3 J_3)}(g) \exp[N_c W(g)], \quad (5.14)$$

$$\langle 0 | \mathcal{B}(g) | B(T = J, T_3 J_3), \pi_a(\mathbf{k}) \rangle = N_c^{\nu_B + \frac{1}{2}} A_{B(T=J, T_3 J_3), \pi_a(\mathbf{k})}(g) \exp[N_c W(g)]. \quad (5.15)$$

Inserting this ansatz into Eq. (5.13), we obtain in the leading order of the $1/N_c$ expansion

$$\begin{aligned} N_c^{1/2} A_{B(T=J, T_3 J_3), \pi_a(\mathbf{k})}(g) &= \frac{i N_c}{2F_\pi} \left(g \tau_a \gamma_5 \cdot \frac{\delta W}{\delta g} \right) A_{B(T=J, T_3 J_3)}(g) \\ &+ \sum_{T' T'_3 J'_3 n} \frac{i}{2F_\pi} G_A^{T' T'_3 J'_3; T T_3 J_3; an} \frac{k^n}{E_\pi} A_{B(T'=J', T'_3 J'_3)}(g). \end{aligned} \quad (5.16)$$

Here we use the short notation

$$\left(g \tau_a \gamma_5 \cdot \frac{\delta W}{\delta g} \right) = \sum_{f_1 f_2 s_1 s_2} \int dy g_{f_1 s_1}(y) (\tau_a)_{f_1 f_2} (\gamma_5)_{s_1 s_2} \frac{\delta W(g)}{\delta g_{f_2 s_2}(y)}. \quad (5.17)$$

Substituting Eq. (4.45) into Eq. (5.16), we find

$$\begin{aligned} & N_c^{1/2} A_{B(T=J, T_3 J_3), \pi_a(\mathbf{k})}(g) \\ &= \frac{i}{2F_\pi} A_0(g) \left[N_c \sqrt{2T+1} \left(g\tau_a \gamma_5 \cdot \frac{\delta W}{\delta g} \right) D_{T_3 J_3}^T(Q(g)) \right. \\ & \quad \left. + \sum_{T' T'_3 J'_3 n} \frac{k^n}{E_\pi} G_A^{T' T'_3 J'_3; T T_3 J_3; an} \sqrt{2T'+1} D_{T'_3 J'_3}^{T'}(Q(g)) \right]. \end{aligned} \quad (5.18)$$

Now we use identity (B5) from Appendix B:

$$\sum_{T' T'_3 J'_3} G_A^{T' T'_3 J'_3; T T_3 J_3; an} \sqrt{2T'+1} D_{T'_3 J'_3}^{T'}(Q(g)) = -3g_A \sqrt{2T+1} D_{an}^{\text{vec}}(i\tau_2 Q(g)) D_{T_3 J_3}^T(Q(g)). \quad (5.19)$$

Function D_{an}^{vec} is defined in Eq. (A8). Inserting Eq. (5.19) into Eq. (5.18), we arrive at

$$\begin{aligned} A_{B(T^B=J^B, T_3^B J_3^B), \pi_a(\mathbf{k})}(g) &= \frac{iN_c^{1/2}}{2F_\pi} A_0(g) D_{T_3^B J_3^B}^{T^B}(Q(g)) \sqrt{2T^B+1} \\ & \quad \times \left[\left(g\tau_a \gamma_5 \cdot \frac{\delta W}{\delta g} \right) - 3 \frac{g_A}{N_c} \frac{k^n}{E_\pi} D_{an}^{\text{vec}}(i\tau_2 Q(g)) \right]. \end{aligned} \quad (5.20)$$

In this equation we have changed the notation:

$$T \rightarrow T^B, J \rightarrow J^B \quad (5.21)$$

in order to emphasize that we deal with the baryon quantum numbers and not with the total isospin and angular momentum of the baryon-pion state.

Note that at large N_c

$$F_\pi = O(N_c^{1/2}), \quad g_A = O(N_c). \quad (5.22)$$

Therefore the ratios $N_c^{1/2}/F_\pi$ and g_A/N_c appearing in Eq. (5.20) have finite large- N_c limits. We can rewrite Eq. (5.20) in terms of the functional $\Xi(g)$ (4.6):

$$\begin{aligned} \Xi_{B(T^B=J^B, T_3^B J_3^B), \pi_a(\mathbf{k})}(g) &= \frac{iN_c^{1/2}}{2F_\pi} D_{T_3^B J_3^B}^{T^B}(Q(g)) \sqrt{2T^B+1} \\ & \quad \times \left[\left(g\tau_a \gamma_5 \cdot \frac{\delta W}{\delta g} \right) - 3 \frac{g_A}{N_c} \frac{k^n}{E_\pi} D_{an}^{\text{vec}}(i\tau_2 Q(g)) \right]. \end{aligned} \quad (5.23)$$

D. Consistency of the large- N_c factorization with the soft-pion theorem

Our aim is to show that the general structure (4.7) of the functional $\Xi(g)$ is consistent with the soft-pion theorem. The main part of this work has been done. We have derived the factorized expression (5.23) for the functional $\Xi_{B\pi}(g)$ describing the baryon-pion state with a soft pion. Our result (5.23) is written for the $B\pi$ states characterized by separate quantum numbers $T^B = J^B, T_3^B J_3^B$ for the baryon and a, \mathbf{k} for the pion:

$$|B(T^B = J^B, T_3^B J_3^B); \pi_a(\mathbf{k})\rangle. \quad (5.24)$$

However, the general expression (4.7) for the functional $\Xi(g)$ is formulated for the eigenstates

$$|TT_3 J J_3 K L, k\rangle, \quad k \equiv |\mathbf{k}| \quad (5.25)$$

of the set of commuting operators (3.44). In addition to quantum numbers (3.44), here we also have the orbital angular momentum L . Since the pion has no spin, the total angular momentum is

$$\mathbf{J} = \mathbf{J}^B + \mathbf{L}. \quad (5.26)$$

Strictly speaking, neither J^B nor L are conserved in the pion-baryon scattering. The precise definition of states (5.25) should be formulated in terms of *in*- and *out*-states, S -matrix, etc., as is usually done in the large- N_c approach to the

meson-baryon scattering [18, 19, 20, 21, 22, 23, 24]. However, for the soft pions the difference between the *in*- and *out*-states can be neglected.

The transition from the basis of states (5.24) to the basis (5.25) is a matter of standard manipulations with the angular momentum which are described in Appendix C. As a result of this change of the basis, one passes from the functionals $\Xi_{B(T^B=J^B, T_3^B J_3^B), \pi_a(\mathbf{k})}(g)$ (5.23) to the functionals $\Xi_{TT_3 J J_3 B, \pi(k, L)}^K(g)$. One can see directly from Eq. (5.23) that this amplitude contains only the waves with $L = 0$ and $L = 1$. It is also easy to check (see Appendix C) that the $L = 0$ component corresponds to the value $K = 1$ whereas the $L = 1$ contribution corresponds to $K = 0$. As a result, in the soft-pion approximation (5.23) we have only two nonzero components of $\Xi_{TT_3 J J_3 B, \pi(k, L)}^K(g)$:

$$\Xi_{TT_3 J J_3 B, \pi(k, L=1)}^{K=0}(g) = -i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \left[\delta_{TJ} \sqrt{2T+1} D_{T_3 J_3}^T(Q(g)) \frac{3g_A}{N_c} \frac{k}{E_\pi} \right], \quad (5.27)$$

$$\begin{aligned} \Xi_{TT_3 J J_3 B, \pi(k, L=0)}^{K=1}(g) &= i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \\ &\times \left[\sqrt{(2T+1)(2J+1)} \delta_{JJ^B} \sum_{T_3' K_3} (-1)^{T+1+J_3} \begin{pmatrix} T & 1 & J \\ T_3' & K_3 & -J_3 \end{pmatrix} D_{T_3 T_3'}^T(Q(g)) \zeta_{K_3}(g) \right]. \end{aligned} \quad (5.28)$$

The last equation contains functional $\zeta_{K_3}(g)$ whose expression via the universal functional $W(g)$ is given by Eq. (C7).

The structure of these results completely agrees with the general representation (4.8). Note that the factors $\xi_{mKK_3}(g)$ of Eq. (4.8) are now associated with the soft pions. In this case the role of index m in $\xi_{mKK_3}(g)$ is played by two pion parameters

$$m = \{L, k\}. \quad (5.29)$$

Comparing Eqs. (5.27) and (5.28) with Eq. (4.8), we find

$$\xi_{\{L=0, k\}, K=1, K_3}(g) = i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \zeta_{K_3}(g), \quad (5.30)$$

$$\xi_{\{L=1, k\}, K=0, K_3=0}(g) = -i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \frac{3g_A}{N_c} \frac{k}{E_\pi}. \quad (5.31)$$

Note that the functional $\xi_{\{L=1, k\}, K=0, K_3}(g)$ is g independent. In principle, this g independence is an artefact of the soft-pion limit. Nevertheless this independence is a good illustration of the warning made in Sec. IV A: The interpretation of functionals $A_B(g)$ as a special representation for baryon-meson states $|B\rangle$ is not quite correct since different states $|B\rangle$ can be mapped to identical functionals $A_B(g)$.

E. Lessons from the soft-pion theorem

The result of our analysis of the soft-pion theorem is represented by the expressions (5.30), (5.31) for the two nonvanishing functionals $\xi_{\{L, k\}KK_3}(g)$ describing the $L = 0$ and $L = 1$ waves of the soft-pion scattering states. The result of the calculation is not of the primary importance for us. What is really significant is that the physics of soft pions is compatible with the general representation (4.6), (4.8) for the functionals $A_B(g)$.

In our analysis we have concentrated on the case of baryon-pion states with one pion. But the factorized structure of the expression (5.23) allows for a straightforward generalization to the multipion case so that the representation (4.6), (4.8) for $A_B(g)$ can be checked also for multipion states.

In this paper, the consistency of the representation (4.8) was checked using two methods:

- spin-flavor symmetry,
- soft-pion theorem.

It is well known that there is a deep connection between the two approaches. The spin-flavor symmetry has an elegant derivation [12, 13, 14] based on the analysis of the soft pion-baryon scattering (strictly speaking, the chiral limit and the existence of the Goldstone bosons are not necessary for the spin-flavor symmetry, this symmetry is a consequence of the large- N_c limit only). Our analysis of the soft-pion theorem for the functional $A_B(g)$ and the traditional derivation [12, 13, 14] of the spin-flavor symmetry from the consistency condition for the pion-baryon scattering have many common points.

VI. CONCLUSIONS

The large- N_c description of the distribution amplitude of baryon B in terms of the generating functional $\Phi_B(g)$ is based on the important property of universality. The exponential behavior $\Phi_B(g) \sim \exp[N_c W(g)]$ is governed by the same functional $W(g)$ for all baryons (and baryon-meson scattering states) with the excitation energies $O(N_c^{-1})$ and $O(N_c^0)$.

However, if one goes beyond the exponential accuracy and considers the preexponential factors $A_B(g)$, one loses the universality: the functional $A_B(g)$ depends on the state B . Although we do not know the explicit expression for $A_B(g)$, we have found that $A_B(g)$ has a simple factorized structure which can be described in terms of elementary excitations and zero mode factors.

Some evidence for the factorization of $A_B(g)$ was observed earlier [11] in the context of the evolution equation and the asymptotic limit of large scales $\mu \rightarrow \infty$. This work provides much more solid arguments based on the spin-flavor symmetry. We have also demonstrated the consistency of the large- N_c factorization of $A_B(g)$ with the soft-pion theorem.

Combining the universality of the exponential large- N_c behavior and the factorized structure of $A_B(g)$, we have derived a simple relation between the generating functionals describing the distribution amplitudes of the nucleon and the Δ resonance.

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APPENDIX A: WIGNER FUNCTIONS

The Wigner functions $D_{m_1 m_2}^j(R)$ for the $SU(2)$ group have the well known properties:

$$D_{m_1 m_2}^j(R_1 R_2) = \sum_{m=-j}^j D_{m_1 m}^j(R_1) D_{m m_2}^j(R_2), \quad (\text{A1})$$

$$\left[D_{m m'}^j(R) \right]^* = (-1)^{m-m'} D_{-m, -m'}^j(R), \quad (\text{A2})$$

$$D_{m m'}^j(i\tau_2) = (-1)^{j-m} \delta_{m, -m'}, \quad (\text{A3})$$

$$D_{m_1 m_2}^j(R^{\text{tr}}) = D_{m_2 m_1}^j(R). \quad (\text{A4})$$

Here R^{tr} stands for the transposed matrix.

The $SU(2)$ invariant measure dR is normalized by the condition

$$\int dR = 1. \quad (\text{A5})$$

Then

$$\int dR \left[D_{m_1 m_2}^j(R) \right]^* D_{m'_1 m'_2}^{j'}(R) = \frac{1}{2j+1} \delta_{jj'} \delta_{m_1 m'_1} \delta_{m_2 m'_2}. \quad (\text{A6})$$

The integral of the product of three D functions can be expressed via $3j$ symbols:

$$\int dR D_{m'_1 m_1}^{j_1}(R) D_{m'_2 m_2}^{j_2}(R) D_{m'_3 m_3}^{j_3}(R) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \quad (\text{A7})$$

m	-1	0	1
a			
1	$i/\sqrt{2}$	0	$-i/\sqrt{2}$
2	$1/\sqrt{2}$	0	$1/\sqrt{2}$
3	0	i	0

TABLE I: Matrix O_{am} .

The standard Wigner function $D_{mm'}^1(R)$ corresponding to the spin $j = 1$ is equivalent to the vector representation $D_{ab}^{\text{vec}}(R)$:

$$D_{ab}^{\text{vec}}(R) = \frac{1}{2} \text{Tr} (\tau_a R \tau_b R^{-1}) , \quad (\text{A8})$$

$$D_{mm'}^1(R) = \sum_{ab} O_{am}^* O_{bm'} D_{ab}^{\text{vec}}(R) . \quad (\text{A9})$$

The matrix O_{am} is given in Table I. This matrix performs the unitary transformation from the spin-one states labeled by the momentum projection $m = 0, \pm 1$ to the basis with vector indices $a = 1, 2, 3$. Matrix O_{am} has the properties

$$OO^+ = 1, \quad (\text{A10})$$

$$O_{am}^* = (-1)^{m+1} O_{a,-m} . \quad (\text{A11})$$

The spherical functions $Y_{lm}(\mathbf{n})$ with $l = 0$ and $l = 1$ have the form

$$Y_{00}(\mathbf{n}) = \frac{1}{\sqrt{4\pi}} , \quad (\text{A12})$$

$$Y_{1m}(\mathbf{n}) = \sqrt{\frac{3}{4\pi}} n^i O_{im} . \quad (\text{A13})$$

The connection between the Clebsh-Gordan coefficients and $3j$ symbols is

$$C_{j_1 m_1 j_2 m_2}^{jm} = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} . \quad (\text{A14})$$

APPENDIX B: AXIAL BARYON COUPLING CONSTANTS AT LARGE N_c

At large N_c the axial baryon couplings $G_A^{T'T_3 J'_3; T T_3 J_3; an}$ appearing in Eq. (5.9) can be expressed in terms of $3j$ symbols:

$$\begin{aligned} G_A^{T'T_3 J'_3; T T_3 J_3; an} &= -3g_A (-1)^{J+J'_3} (-1)^{T+T'_3} \sqrt{(2T'+1)(2T+1)} \\ &\times \sum_{tm} O_{nm}^* O_{at} \begin{pmatrix} T & 1 & T' \\ T_3 & t & -T'_3 \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ J_3 & m & -J'_3 \end{pmatrix} . \end{aligned} \quad (\text{B1})$$

Here

$$J' = T' , \quad J = T , \quad (\text{B2})$$

g_A is the axial constant of nucleon, and matrix O_{nm} is defined in Table I.

In the context of the spin-flavor symmetry of large- N_c QCD, Eq. (B1) is discussed in Refs. [14, 16]. In the Skyrme model and other chiral models expression (B1) appears via the integral over the $SU(2)$ matrices R [27, 36]

$$G_A^{T'T'_3J'_3;TT_3J_3;an} = -3g_A \int dR \left[\psi_{T'_3J'_3}^{T'}(R) \right]^* D_{an}^{\text{vec}}(R) \psi_{T_3J_3}^T(R), \quad (\text{B3})$$

where $\psi_{T_3J_3}^T(R)$ are the rotational wave functions of the soliton

$$\psi_{T_3J_3}^{T=J}(R) = (-1)^{T+T_3} \sqrt{2T+1} D_{-T_3, J_3}^{T=J}(R), \quad (\text{B4})$$

and function $D_{an}^{\text{vec}}(R)$ is defined by Eq. (A8). The normalization coefficient in Eq. (B1) is chosen so that in the nucleon case $T = T' = 1/2$ one obtains Eq. (5.10).

It is straightforward to show that

$$\sum_{T'J'_3T'_3} G_A^{T'T'_3J'_3;TT_3J_3;an} \sqrt{2T'+1} D_{T'_3J'_3}^{T'}(R) = -3g_A \sqrt{2T+1} D_{an}^{\text{vec}}(i\tau_2 R) D_{T_3J_3}^T(R). \quad (\text{B5})$$

This equality also holds for arbitrary $SL(2, C)$ matrices R if one replaces the $SU(2)$ Wigner functions D^T by their $SL(2, C)$ generalization $D^{(T, \emptyset)}$.

APPENDIX C: SOFT-PION FACTORS

In this appendix, we construct the baryon-meson states (5.25) from the states (5.24) and compute the corresponding functionals $\Xi_{TT_3JJ_3B, \pi(k, L)}^K(g)$ (5.27) and (5.28). As a first step we build the eigenstates of operators

$$T^2, T_3, J^2, J_3, L^2, (J^B)^2 = (T^B)^2, \quad (\text{C1})$$

where T is the total isospin, J is the total angular momentum, and L is the orbital momentum. In order to avoid confusion, we mark the baryon quantum numbers $J^B = T^B$ with the label B . The standard rules of the addition of angular momenta yield:

$$\begin{aligned} & |TT_3JJ_3, B(T^B = J^B), \pi(k, L)\rangle \\ &= \sum_{T_3^B t} \sum_{J_3^B m} C_{T^B T_3^B, 1t}^{TT_3} C_{J^B J_3^B, Lm}^{JJ_3} \sum_a \int d^2\mathbf{n} Y_{Lm}(\mathbf{n}) O_{at} |B(T^B = J^B, T_3^B J_3^B), \pi_a(k\mathbf{n})\rangle. \end{aligned} \quad (\text{C2})$$

Here O_{at} is the matrix from Table I. This matrix converts the states $|\pi_a\rangle$ with indices $a = 1, 2, 3$ (associated with the Pauli matrices τ_a) to the eigenstates of the isospin $t = 0, \pm 1$. The RHS of Eq. (C2) contains the integration over the direction \mathbf{n} of the pion momentum $k\mathbf{n}$. Combining Eq. (C2) with Eq. (5.20), we find

$$\begin{aligned} A_{TT_3JJ_3B(T^B=J^B), \pi(k, L)}(g) &= \int d^2\mathbf{n} \sum_{T_3^B t} \sum_{J_3^B m} Y_{Lm}(\mathbf{n}) C_{T^B T_3^B, 1t}^{TT_3} C_{J^B J_3^B, Lm}^{JJ_3} \frac{iN_c^{1/2}}{2F_\pi} A_0(g) D_{T_3^B J_3^B}^{T^B}(Q(g)) \\ &\times \sqrt{2T^B+1} \left[\eta_t(g) - 3 \frac{g_A}{N_c} \frac{k}{E_\pi} \sum_{aj} n^j O_{at} D_{aj}^{\text{vec}}(i\tau_2 Q(g)) \right], \end{aligned} \quad (\text{C3})$$

where

$$\eta_t(g) = \sum_a O_{at} \left[g\tau_a \gamma_5 \cdot \frac{\delta W(g)}{\delta g} \right]. \quad (\text{C4})$$

Next we integrate over \mathbf{n} using expressions (A12) and (A13) for $Y_{Lm}(\mathbf{n})$. Applying Eqs. (A3), (A9), and (A11), we find

$$\begin{aligned} A_{TT_3JJ_3B(T^B=J^B), \pi(k, L)}(g) &= i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} A_0(g) \\ &\times \left[\delta_{L0} \sqrt{2J+1} \delta_{JJ^B} \sum_{T_3^B t} C_{T^B T_3^B, 1t}^{TT_3} D_{T_3^B J_3^B}^{T^B}(Q(g)) \eta_t(g) - \sqrt{2T^B+1} \delta_{L1} \delta_{TJ} D_{T_3^B J_3^B}^J(Q(g)) \sqrt{3} \frac{g_A}{N_c} \frac{k}{E_\pi} \right]. \end{aligned} \quad (\text{C5})$$

Let us introduce the notation

$$\zeta_{K_3}(g) = (-1)^{1+K_3} \sum_t D_{-K_3,t}^1(Q^{-1}(g)) \eta_t(g). \quad (C6)$$

Inserting Eq. (C4) into Eq. (C6), we obtain

$$\zeta_{K_3}(g) = (-1)^{1+K_3} \sum_t D_{-K_3,t}^1(Q^{-1}(g)) \sum_a O_{at} \left[g\tau_a \gamma_5 \cdot \frac{\delta W(g)}{\delta g} \right]. \quad (C7)$$

Now we find from Eq. (C5) using Eq. (C6)

$$\begin{aligned} A_{TT_3JJ_3B(T^B=J^B),\pi(k,L)}(g) &= i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} A_0(g) \\ &\times \left[\delta_{L0} \sqrt{2J+1} \delta_{JJ^B} \sum_{T'_3 K_3} (-1)^{1-K_3} C_{T^B J_3, 1, -K_3}^{TT'_3} D_{T_3 T'_3}^T(Q(g)) \zeta_{K_3}(g) - \delta_{L1} \delta_{TJ} \sqrt{2T^B+1} D_{T_3 J_3}^J(Q(g)) \sqrt{3} \frac{g_A}{N_c} \frac{k}{E_\pi} \right]. \end{aligned} \quad (C8)$$

In terms of the functional (4.6)

$$\Xi_{TT_3JJ_3B(T^B=J^B),\pi(k,L)}(g) = [A_0(g)]^{-1} A_{TT_3JJ_3B(T^B=J^B),\pi(k,L)}(g) \quad (C9)$$

we obtain from Eq. (C8)

$$\begin{aligned} \Xi_{TT_3JJ_3B(T^B=J^B),\pi(k,L)}(g) &= i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \\ &\times \left[\delta_{L0} \sqrt{(2T+1)(2J+1)} \delta_{JJ^B} \sum_{T'_3 K_3} (-1)^{J+J_3} \begin{pmatrix} T & 1 & J \\ T'_3 & K_3 & -J_3 \end{pmatrix} D_{T_3 T'_3}^T(Q(g)) \zeta_{K_3}(g) \right. \\ &\quad \left. - \delta_{L1} \delta_{TJ} \sqrt{2T^B+1} D_{T_3 J_3}^T(Q(g)) \sqrt{3} \frac{g_A}{N_c} \frac{k}{E_\pi} \right]. \end{aligned} \quad (C10)$$

This expression is computed for the state (C2), which is not an eigenstate of K . Now we construct the eigenstates of K :

$$|TT_3JJ_3KB, \pi(k, L)\rangle = \sum_{T^B} |TT_3JJ_3B(T^B=J^B), \pi(k, L)\rangle \langle T^B=J^B|K\rangle_{JTL}. \quad (C11)$$

The expression for the matrix elements $\langle T^B=J^B|K\rangle_{JTL}$ in terms of $6j$ symbols can be read from Ref. [18]:

$$\langle T^B=J^B|K\rangle_{JTL} = (-1)^{L+T^B+J} \sqrt{(2T^B+1)(2K+1)} \left\{ \begin{matrix} K & T & J \\ T^B & L & 1 \end{matrix} \right\}. \quad (C12)$$

This leads to the expressions

$$\langle T^B=J^B|K=1\rangle_{JT,L=0} = (-1)^{T+1-J} \delta_{T^B J}, \quad (C13)$$

$$\langle T^B=J^B|K=0\rangle_{JT,L=1} = \sqrt{\frac{2T^B+1}{3(2T+1)}} \delta_{TJ}. \quad (C14)$$

Now we derive from Eqs. (C10), (C11), and (C14):

$$\Xi_{TT_3JJ_3B,\pi(k,L=1)}^{K=0}(g) = -i\sqrt{\pi} \frac{N_c^{1/2}}{F_\pi} \delta_{TJ} D_{T_3 J_3}^T[Q(g)] \frac{g_A}{N_c} \frac{k}{E_\pi} \sum_{T^B} \frac{2T^B+1}{\sqrt{2T+1}}. \quad (C15)$$

The summation over T^B is restricted by the condition

$$|T-1| \leq T^B \leq T+1. \quad (C16)$$

At $T > 1/2$ this leads to

$$\sum_{T^B} \frac{2T^B + 1}{\sqrt{2T + 1}} = \sum_{T^B=T-1}^{T+1} \frac{2T^B + 1}{\sqrt{2T + 1}} = 3\sqrt{2T + 1}. \quad (\text{C17})$$

Actually we have

$$\sum_{T^B} \frac{2T^B + 1}{\sqrt{2T + 1}} = 3\sqrt{2T + 1}. \quad (\text{C18})$$

for any T (at $T = 0$ the only possible value is $T^B = 1$).

Inserting Eq. (C18) into Eq. (C15), we obtain Eq. (5.27). Similarly one derives Eq. (5.28) from Eqs. (C10), (C11), and (C13).

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